

E Complex Numbers

- Use the imaginary unit i to write complex numbers and to add, subtract, and multiply complex numbers.
- Find complex solutions of quadratic equations.
- Write the trigonometric forms of complex numbers.
- Find powers and n th roots of complex numbers.

Operations with Complex Numbers

Some equations have no real solutions. For instance, the quadratic equation

$$x^2 + 1 = 0 \quad \text{Equation with no real solution}$$

has no real solution because there is no real number x that can be squared to produce -1 . To overcome this deficiency, mathematicians created an expanded system of numbers using the **imaginary unit** i , defined as

$$i = \sqrt{-1} \quad \text{Imaginary unit}$$

where $i^2 = -1$. By adding real numbers to real multiples of this imaginary unit, you obtain the set of **complex numbers**. Each complex number can be written in the **standard form** $a + bi$. The real number a is called the **real part** of the **complex number** $a + bi$, and the number bi (where b is a real number) is called the **imaginary part** of the complex number.

Definition of a Complex Number

For real numbers a and b , the number

$$a + bi$$

is a **complex number**. If $b \neq 0$, then $a + bi$ is called an **imaginary number**. A number of the form bi , where $b \neq 0$, is called a **pure imaginary number**.

To add (or subtract) two complex numbers, you add (or subtract) the real and imaginary parts of the numbers separately.

Addition and Subtraction of Complex Numbers

If $a + bi$ and $c + di$ are two complex numbers written in standard form, then their sum and difference are defined as follows.

$$\text{Sum: } (a + bi) + (c + di) = (a + c) + (b + d)i$$

$$\text{Difference: } (a + bi) - (c + di) = (a - c) + (b - d)i$$

The **additive identity** in the complex number system is zero (the same as in the real number system). Furthermore, the **additive inverse** of the complex number $a + bi$ is

$$-(a + bi) = -a - bi. \quad \text{Additive inverse}$$

So, you have

$$(a + bi) + (-a - bi) = 0 + 0i = 0.$$

EXAMPLE 1 Adding and Subtracting Complex Numbers

- a. $(3 - i) + (2 + 3i) = 3 - i + 2 + 3i$ Remove parentheses.
 $= 3 + 2 - i + 3i$ Group like terms.
 $= (3 + 2) + (-1 + 3)i$
 $= 5 + 2i$ Write in standard form.
- b. $2i + (-4 - 2i) = 2i - 4 - 2i$ Remove parentheses.
 $= -4 + 2i - 2i$ Group like terms.
 $= -4$ Write in standard form.
- c. $3 - (-2 + 3i) + (-5 + i) = 3 + 2 - 3i - 5 + i$
 $= 3 + 2 - 5 - 3i + i$
 $= 0 - 2i$
 $= -2i$

In Example 1(b), notice that the sum of two complex numbers can be a real number.

Many of the properties of real numbers are valid for complex numbers as well. Here are some examples.

Associative Properties of Addition and Multiplication
Commutative Properties of Addition and Multiplication
Distributive Property of Multiplication over Addition

Notice how these properties are used when two complex numbers are multiplied.

$$\begin{aligned} (a + bi)(c + di) &= a(c + di) + bi(c + di) && \text{Distributive Property} \\ &= ac + (ad)i + (bc)i + (bd)i^2 && \text{Distributive Property} \\ &= ac + (ad)i + (bc)i + (bd)(-1) && i^2 = -1 \\ &= ac - bd + (ad)i + (bc)i && \text{Commutative Property} \\ &= (ac - bd) + (ad + bc)i && \text{Associative Property} \end{aligned}$$

The procedure above is similar to multiplying two polynomials and combining like terms, as in the FOIL method.

•• **REMARK** Rather than trying to memorize the multiplication rule at the right, you can simply remember how the Distributive Property is used to multiply two complex numbers. The procedure is similar to multiplying two polynomials and combining like terms.



EXAMPLE 2**Multiplying Complex Numbers**

a. $(3 + 2i)(3 - 2i) = 3(3 - 2i) + 2i(3 - 2i)$ Distributive Property
 $= 9 - 6i + 6i - 4i^2$ Distributive Property
 $= 9 - 6i + 6i - 4(-1)$ $i^2 = -1$
 $= 9 + 4$ Simplify.
 $= 13$ Write in standard form.

b. $(3 + 2i)^2 = (3 + 2i)(3 + 2i)$ Square of a binomial
 $= 3(3 + 2i) + 2i(3 + 2i)$ Distributive Property
 $= 9 + 6i + 6i + 4i^2$ Distributive Property
 $= 9 + 6i + 6i + 4(-1)$ $i^2 = -1$
 $= 9 + 12i - 4$ Simplify.
 $= 5 + 12i$ Write in standard form. 

In Example 2(a), notice that the product of two complex numbers can be a real number. This occurs with pairs of complex numbers of the form $a + bi$ and $a - bi$, called **complex conjugates**.

$$\begin{aligned}(a + bi)(a - bi) &= a^2 - abi + abi - b^2i^2 \\ &= a^2 - b^2(-1) \\ &= a^2 + b^2\end{aligned}$$

To write the quotient of $a + bi$ and $c + di$ in standard form, where c and d are not both zero, multiply the numerator and denominator by the complex conjugate of the denominator to obtain

$$\begin{aligned}\frac{a + bi}{c + di} &= \frac{a + bi}{c + di} \left(\frac{c - di}{c - di} \right) && \text{Multiply numerator and denominator} \\ &= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2} && \text{by complex conjugate of denominator.} \\ & && \text{Write in standard form.}\end{aligned}$$

EXAMPLE 3**Writing Complex Numbers in Standard Form**

$$\begin{aligned}\frac{2 + 3i}{4 - 2i} &= \frac{2 + 3i}{4 - 2i} \left(\frac{4 + 2i}{4 + 2i} \right) && \text{Multiply numerator and denominator} \\ &= \frac{8 + 4i + 12i + 6i^2}{16 - 4i^2} && \text{by complex conjugate of denominator.} \\ &= \frac{8 - 6 + 16i}{16 + 4} && \text{Expand.} \\ &= \frac{2 + 16i}{20} && \text{Simplify.} \\ &= \frac{1}{10} + \frac{4}{5}i && \text{Write in standard form. $$

Complex Solutions of Quadratic Equations

When using the Quadratic Formula to solve a quadratic equation, you often obtain a result such as $\sqrt{-3}$, which you know is not a real number. By factoring out $i = \sqrt{-1}$, you can write this number in standard form.

$$\sqrt{-3} = \sqrt{3(-1)} = \sqrt{3}\sqrt{-1} = \sqrt{3}i$$

The number $\sqrt{3}i$ is called the *principal square root* of -3 .

REMARK The definition of principal square root uses the rule

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

for $a > 0$ and $b < 0$. This rule is not valid when *both* a and b are negative.

For example,

$$\begin{aligned}\sqrt{-5}\sqrt{-5} &= \sqrt{5(-1)}\sqrt{5(-1)} \\ &= \sqrt{5}i\sqrt{5}i \\ &= \sqrt{25}i^2 \\ &= 5i^2 \\ &= -5\end{aligned}$$

whereas

$$\sqrt{(-5)(-5)} = \sqrt{25} = 5.$$

To avoid problems with multiplying square roots of negative numbers, be sure to convert to standard form *before* multiplying.

Principal Square Root of a Negative Number

If a is a positive number, then the **principal square root** of the negative number $-a$ is defined as

$$\sqrt{-a} = \sqrt{a}i.$$

EXAMPLE 4 Writing Complex Numbers in Standard Form

$$\begin{aligned}\text{a. } \sqrt{-3}\sqrt{-12} &= \sqrt{3}i\sqrt{12}i \\ &= \sqrt{36}i^2 \\ &= 6(-1) \\ &= -6 \\ \text{b. } \sqrt{-48} - \sqrt{-27} &= \sqrt{48}i - \sqrt{27}i \\ &= 4\sqrt{3}i - 3\sqrt{3}i \\ &= \sqrt{3}i \\ \text{c. } (-1 + \sqrt{-3})^2 &= (-1 + \sqrt{3}i)^2 \\ &= (-1)^2 - 2\sqrt{3}i + (\sqrt{3})^2(i^2) \\ &= 1 - 2\sqrt{3}i + 3(-1) \\ &= -2 - 2\sqrt{3}i\end{aligned}$$

EXAMPLE 5 Complex Solutions of a Quadratic Equation

Solve $3x^2 - 2x + 5 = 0$.

Solution

$$\begin{aligned}x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(5)}}{2(3)} \\ &= \frac{2 \pm \sqrt{-56}}{6} \\ &= \frac{2 \pm 2\sqrt{14}i}{6} \\ &= \frac{1}{3} \pm \frac{\sqrt{14}}{3}i\end{aligned}$$

Quadratic Formula

Simplify.

Write $\sqrt{-56}$ in standard form.

Write in standard form.

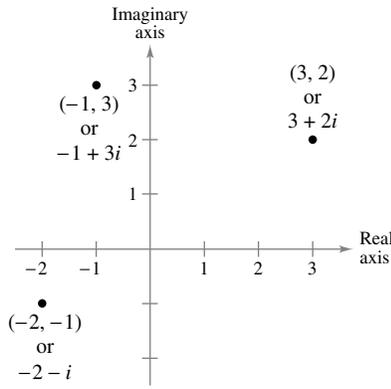


Figure E.1

Trigonometric Form of a Complex Number

Just as real numbers can be represented by points on the real number line, you can represent a complex number

$$z = a + bi$$

as the point (a, b) in a coordinate plane (the **complex plane**). The horizontal axis is called the **real axis** and the vertical axis is called the **imaginary axis**, as shown in Figure E.1.

The **absolute value** of a complex number $a + bi$ is defined as the distance between the origin $(0, 0)$ and the point (a, b) .

The Absolute Value of a Complex Number

The **absolute value** of the complex number $z = a + bi$ is given by

$$|a + bi| = \sqrt{a^2 + b^2}.$$

When the complex number $a + bi$ is a real number (that is, $b = 0$), this definition agrees with that given for the absolute value of a real number.

$$|a + 0i| = \sqrt{a^2 + 0^2} = |a|$$

To work effectively with *powers* and *roots* of complex numbers, it is helpful to write complex numbers in **trigonometric form**. In Figure E.2, consider the nonzero complex number $a + bi$. By letting θ be the angle from the positive real axis (measured counterclockwise) to the line segment connecting the origin and the point (a, b) , you can write

$$a = r \cos \theta \quad \text{and} \quad b = r \sin \theta$$

where $r = \sqrt{a^2 + b^2}$. Consequently, you have

$$a + bi = (r \cos \theta) + (r \sin \theta)i$$

from which you can obtain the **trigonometric form of a complex number**.

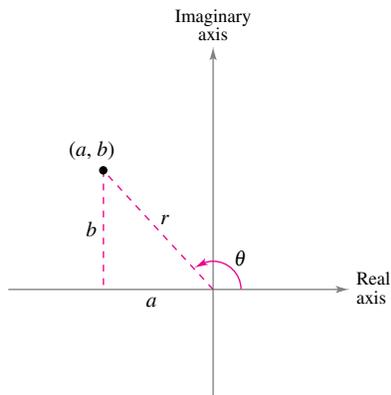


Figure E.2

Trigonometric Form of a Complex Number

The **trigonometric form** of the complex number $z = a + bi$ is given by

$$z = r(\cos \theta + i \sin \theta)$$

where $a = r \cos \theta$, $b = r \sin \theta$, $r = \sqrt{a^2 + b^2}$, and $\tan \theta = b/a$. The number r is the **modulus** of z , and θ is called an **argument** of z .

The trigonometric form of a complex number is also called the **polar form**. Because there are infinitely many choices for θ , the trigonometric form of a complex number is not unique. Normally, θ is restricted to the interval $0 \leq \theta < 2\pi$, although on occasion it is convenient to use $\theta < 0$.

EXAMPLE 6 Trigonometric Form of a Complex Number

Write the complex number $z = -2 - 2\sqrt{3}i$ in trigonometric form.

Solution The absolute value of z is

$$r = |-2 - 2\sqrt{3}i| = \sqrt{(-2)^2 + (-2\sqrt{3})^2} = \sqrt{16} = 4$$

and the angle θ is given by

$$\begin{aligned}\tan \theta &= \frac{b}{a} \\ &= \frac{-2\sqrt{3}}{-2} \\ &= \sqrt{3}.\end{aligned}$$

Because $\tan(\pi/3) = \sqrt{3}$ and because $z = -2 - 2\sqrt{3}i$ lies in Quadrant III, choose θ to be $\theta = \pi + \pi/3 = 4\pi/3$. So, the trigonometric form is

$$\begin{aligned}z &= r(\cos \theta + i \sin \theta) \\ &= 4\left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}\right).\end{aligned}$$

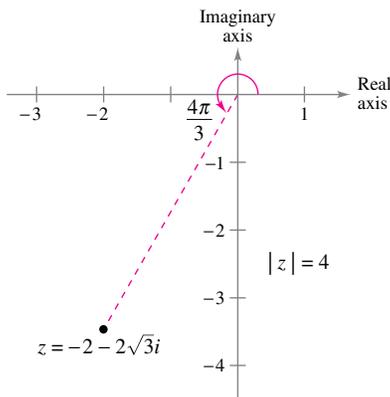


Figure E.3

See Figure E.3. ■

The trigonometric form adapts nicely to multiplication and division of complex numbers. Consider the two complex numbers

$$z_1 = r_1(\cos \theta_1 + i \sin \theta_1) \quad \text{and} \quad z_2 = r_2(\cos \theta_2 + i \sin \theta_2).$$

The product of z_1 and z_2 is

$$\begin{aligned}z_1 z_2 &= r_1 r_2 (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) \\ &= r_1 r_2 [(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)].\end{aligned}$$

Using the sum and difference formulas for cosine and sine, you can rewrite this equation as

$$z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)].$$

This establishes the first part of the rule shown below. The second part is left for you to verify (see Exercise 109).

Product and Quotient of Two Complex Numbers

Let $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$ be complex numbers.

$$z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)] \quad \text{Product}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)], \quad z_2 \neq 0 \quad \text{Quotient}$$

Note that this rule says that to multiply two complex numbers, you multiply moduli and add arguments, whereas to divide two complex numbers, you divide moduli and subtract arguments.

EXAMPLE 7 Multiplying Complex Numbers

Find the product z_1z_2 of the complex numbers.

$$z_1 = 2\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right), \quad z_2 = 8\left(\cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6}\right)$$

Solution

$$\begin{aligned} z_1z_2 &= 2\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right) \cdot 8\left(\cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6}\right) \\ &= 16\left[\cos\left(\frac{2\pi}{3} + \frac{11\pi}{6}\right) + i \sin\left(\frac{2\pi}{3} + \frac{11\pi}{6}\right)\right] && \text{Multiply moduli and} \\ & && \text{add arguments.} \\ &= 16\left[\cos \frac{5\pi}{2} + i \sin \frac{5\pi}{2}\right] \\ &= 16\left[\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right] && \frac{5\pi}{2} \text{ and } \frac{\pi}{2} \text{ are coterminal.} \\ &= 16[0 + i(1)] \\ &= 16i \end{aligned}$$

Check this result by first converting to the standard forms $z_1 = -1 + \sqrt{3}i$ and $z_2 = 4\sqrt{3} - 4i$ and then multiplying algebraically.

EXAMPLE 8 Dividing Complex Numbers

Find the quotient z_1/z_2 of the complex numbers.

$$z_1 = 24(\cos 300^\circ + i \sin 300^\circ), \quad z_2 = 8(\cos 75^\circ + i \sin 75^\circ)$$

Solution

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{24(\cos 300^\circ + i \sin 300^\circ)}{8(\cos 75^\circ + i \sin 75^\circ)} \\ &= \frac{24}{8}[\cos(300^\circ - 75^\circ) + i \sin(300^\circ - 75^\circ)] && \text{Divide moduli and} \\ & && \text{subtract arguments.} \\ &= 3[\cos 225^\circ + i \sin 225^\circ] \\ &= 3\left[\left(-\frac{\sqrt{2}}{2}\right) + i\left(-\frac{\sqrt{2}}{2}\right)\right] \\ &= -\frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}i \end{aligned}$$

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Powers and Roots of Complex Numbers

To raise a complex number to a power, consider repeated use of the multiplication rule.

$$\begin{aligned} z &= r(\cos \theta + i \sin \theta) \\ z^2 &= r^2(\cos 2\theta + i \sin 2\theta) \\ z^3 &= r^3(\cos 3\theta + i \sin 3\theta) \\ &\vdots \end{aligned}$$

This pattern leads to the next theorem, which is named after the French mathematician Abraham DeMoivre (1667–1754).

THEOREM E.1 DeMoivre's Theorem

If $z = r(\cos \theta + i \sin \theta)$ is a complex number and n is a positive integer, then

$$z^n = [r(\cos \theta + i \sin \theta)]^n = r^n(\cos n\theta + i \sin n\theta).$$

EXAMPLE 9 Finding Powers of a Complex Number

Use DeMoivre's Theorem to find $(-1 + \sqrt{3}i)^{12}$.

Solution First convert the complex number to trigonometric form.

$$-1 + \sqrt{3}i = 2\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right)$$

Then, by DeMoivre's Theorem, you have

$$\begin{aligned} (-1 + \sqrt{3}i)^{12} &= \left[2\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right)\right]^{12} \\ &= 2^{12} \left[\cos\left(12 \cdot \frac{2\pi}{3}\right) + i \sin\left(12 \cdot \frac{2\pi}{3}\right)\right] \\ &= 4096(\cos 8\pi + i \sin 8\pi) \\ &= 4096. \end{aligned}$$

Recall that a consequence of the Fundamental Theorem of Algebra is that a polynomial equation of degree n has n solutions in the complex number system. Each solution is an n th root of the equation. The n th root of a complex number is defined below.

Definition of n th Root of a Complex Number

The complex number $u = a + bi$ is an n th root of the complex number z when

$$z = u^n = (a + bi)^n.$$

To find a formula for an n th root of a complex number, let u be an n th root of z , where

$$u = s(\cos \beta + i \sin \beta) \quad \text{and} \quad z = r(\cos \theta + i \sin \theta).$$

By DeMoivre's Theorem and the fact that $u^n = z$, you have

$$s^n(\cos n\beta + i \sin n\beta) = r(\cos \theta + i \sin \theta).$$

Taking the absolute value of each side of this equation, it follows that $s^n = r$. Substituting r for s^n in the previous equation and dividing by r , you get

$$\cos n\beta + i \sin n\beta = \cos \theta + i \sin \theta.$$

So, it follows that

$$\cos n\beta = \cos \theta \quad \text{and} \quad \sin n\beta = \sin \theta.$$

Because both sine and cosine have a period of 2π , these last two equations have solutions if and only if the angles differ by a multiple of 2π . Consequently, there must exist an integer k such that

$$\begin{aligned} n\beta &= \theta + 2\pi k \\ \beta &= \frac{\theta + 2\pi k}{n}. \end{aligned}$$

By substituting this value for β into the trigonometric form of u , you get the result stated in the next theorem.

THEOREM E.2 n th Roots of a Complex Number

For a positive integer n , the complex number $z = r(\cos \theta + i \sin \theta)$ has exactly n distinct n th roots given by

$$\sqrt[n]{r} \left(\cos \frac{\theta + 2\pi k}{n} + i \sin \frac{\theta + 2\pi k}{n} \right)$$

where $k = 0, 1, 2, \dots, n - 1$.

For $k > n - 1$, the roots begin to repeat. For instance, when $k = n$, the angle

$$\frac{\theta + 2\pi n}{n} = \frac{\theta}{n} + 2\pi$$

is coterminal with θ/n , which is also obtained when $k = 0$.

The formula for the n th roots of a complex number z has a nice geometric interpretation, as shown in Figure E.4. Note that because the n th roots of z all have the same magnitude $\sqrt[n]{r}$, they all lie on a circle of radius $\sqrt[n]{r}$ with center at the origin. Furthermore, because successive n th roots have arguments that differ by $2\pi/n$, the n roots are equally spaced along the circle.

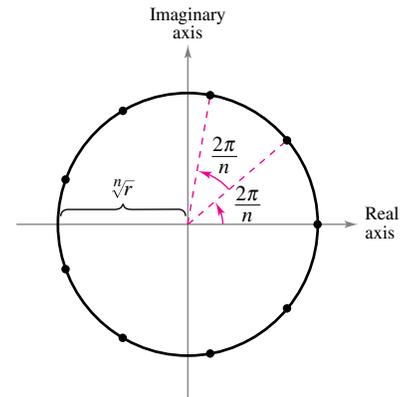


Figure E.4

EXAMPLE 10 Finding the n th Roots of a Complex Number

Find the three cube roots of $z = -2 + 2i$.

Solution Because z lies in Quadrant II, the trigonometric form for z is

$$z = -2 + 2i = \sqrt{8}(\cos 135^\circ + i \sin 135^\circ).$$

By the formula for n th roots, the cube roots have the form

$$\sqrt[3]{8} \left(\cos \frac{135^\circ + 360^\circ k}{3} + i \sin \frac{135^\circ + 360^\circ k}{3} \right).$$

Finally, for $k = 0, 1,$ and $2,$ you obtain the roots

$$\begin{aligned} \sqrt{2}(\cos 45^\circ + i \sin 45^\circ) &= 1 + i \\ \sqrt{2}(\cos 165^\circ + i \sin 165^\circ) &\approx -1.3660 + 0.3660i \\ \sqrt{2}(\cos 285^\circ + i \sin 285^\circ) &\approx 0.3660 - 1.3660i. \end{aligned}$$

E Exercises

Performing Operations In Exercises 1–24, perform the operation and write the result in standard form.

- $(5 + i) + (6 - 2i)$
- $(13 - 2i) + (-5 + 6i)$
- $(8 - i) - (4 - i)$
- $(3 + 2i) - (6 + 13i)$
- $(-2 + \sqrt{-8}) + (5 - \sqrt{-50})$
- $(8 + \sqrt{-18}) - (4 + 3\sqrt{2}i)$
- $13i - (14 - 7i)$
- $22 + (-5 + 8i) + 10i$
- $-\left(\frac{3}{2} + \frac{5}{2}i\right) + \left(\frac{5}{3} + \frac{11}{3}i\right)$
- $(1.6 + 3.2i) + (-5.8 + 4.3i)$
- $\sqrt{-6} \cdot \sqrt{-2}$
- $\sqrt{-5} \cdot \sqrt{-10}$
- $(\sqrt{-10})^2$
- $(\sqrt{-75})^2$
- $(1 + i)(3 - 2i)$
- $(6 - 2i)(2 - 3i)$
- $6i(5 - 2i)$
- $-8i(9 + 4i)$
- $(\sqrt{14} + \sqrt{10}i)(\sqrt{14} - \sqrt{10}i)$
- $(3 + \sqrt{-5})(7 - \sqrt{-10})$
- $(4 + 5i)^2$
- $(2 - 3i)^2$
- $(2 + 3i)^2 + (2 - 3i)^2$
- $(1 - 2i)^2 - (1 + 2i)^2$

Writing a Complex Conjugate In Exercises 25–32, write the complex conjugate of the complex number. Then multiply the number by its complex conjugate.

- | | |
|----------------------|----------------------|
| 25. $5 + 3i$ | 26. $9 - 12i$ |
| 27. $-2 - \sqrt{5}i$ | 28. $-4 + \sqrt{2}i$ |
| 29. $20i$ | 30. $\sqrt{-15}$ |
| 31. $\sqrt{8}$ | 32. $1 + \sqrt{8}$ |

Writing in Standard Form In Exercises 33–42, write the quotient in standard form.

- | | |
|----------------------------|-----------------------------------|
| 33. $\frac{6}{i}$ | 34. $-\frac{10}{2i}$ |
| 35. $\frac{4}{4 - 5i}$ | 36. $\frac{3}{1 - i}$ |
| 37. $\frac{2 + i}{2 - i}$ | 38. $\frac{8 - 7i}{1 - 2i}$ |
| 39. $\frac{6 - 7i}{i}$ | 40. $\frac{8 + 20i}{2i}$ |
| 41. $\frac{1}{(4 - 5i)^2}$ | 42. $\frac{(2 - 3i)(5i)}{2 + 3i}$ |

Performing Operations In Exercises 43–46, perform the operation and write the result in standard form.

- | | |
|--|--|
| 43. $\frac{2}{1 + i} - \frac{3}{1 - i}$ | 44. $\frac{2i}{2 + i} + \frac{5}{2 - i}$ |
| 45. $\frac{i}{3 - 2i} + \frac{2i}{3 + 8i}$ | 46. $\frac{1 + i}{i} - \frac{3}{4 - i}$ |

Using the Quadratic Formula In Exercises 47–54, use the Quadratic Formula to solve the quadratic equation.

47. $x^2 - 2x + 2 = 0$
 48. $x^2 + 6x + 10 = 0$
 49. $4x^2 + 16x + 17 = 0$
 50. $9x^2 - 6x + 37 = 0$
 51. $4x^2 + 16x + 15 = 0$
 52. $9x^2 - 6x - 35 = 0$
 53. $16t^2 - 4t + 3 = 0$
 54. $5s^2 + 6s + 3 = 0$

Writing in Standard Form In Exercises 55–62, simplify the complex number and write it in standard form.

55. $-6i^3 + i^2$
 56. $4i^2 - 2i^3$
 57. $-5i^5$
 58. $(-i)^3$
 59. $(\sqrt{-75})^3$
 60. $(\sqrt{-2})^6$
 61. $\frac{1}{i^3}$
 62. $\frac{1}{(2i)^3}$

Absolute Value of a Complex Number In Exercises 63–68, plot the complex number and find its absolute value.

63. $-5i$ 64. -5
 65. $-4 + 4i$ 66. $5 - 12i$
 67. $6 - 7i$ 68. $-8 + 3i$

Writing in Trigonometric Form In Exercises 69–76, represent the complex number graphically and find the trigonometric form of the number.

69. $3 - 3i$ 70. $2 + 2i$
 71. $\sqrt{3} + i$ 72. $-1 + \sqrt{3}i$
 73. $-2(1 + \sqrt{3}i)$ 74. $\frac{5}{2}(\sqrt{3} - i)$
 75. $6i$ 76. 4

Writing in Standard Form In Exercises 77–82, represent the complex number graphically and find the standard form of the number.

77. $2(\cos 150^\circ + i \sin 150^\circ)$
 78. $5(\cos 135^\circ + i \sin 135^\circ)$
 79. $\frac{3}{2}(\cos 300^\circ + i \sin 300^\circ)$
 80. $\frac{3}{4}(\cos 315^\circ + i \sin 315^\circ)$
 81. $3.75\left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\right)$
 82. $8\left(\cos \frac{\pi}{12} + i \sin \frac{\pi}{12}\right)$

Performing Operations In Exercises 83–86, perform the operation and leave the result in trigonometric form.

83. $\left[3\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)\right]\left[4\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)\right]$
 84. $\left[\frac{3}{2}\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)\right]\left[6\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)\right]$
 85. $\left[\frac{5}{3}(\cos 140^\circ + i \sin 140^\circ)\right]\left[\frac{2}{3}(\cos 60^\circ + i \sin 60^\circ)\right]$
 86. $\frac{\cos(5\pi/3) + i \sin(5\pi/3)}{\cos \pi + i \sin \pi}$

Using DeMoivre's Theorem In Exercises 87–94, use DeMoivre's Theorem to find the indicated power of the complex number. Write the result in standard form.

87. $(1 + i)^5$
 88. $(2 + 2i)^6$
 89. $(-1 + i)^{10}$
 90. $(1 - i)^{12}$
 91. $2(\sqrt{3} + i)^7$
 92. $4(1 - \sqrt{3}i)^3$
 93. $\left(\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4}\right)^{10}$
 94. $\left[2\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)\right]^8$

Finding n th Roots In Exercises 95–100, (a) use Theorem E.2 to find the indicated roots of the complex number, (b) represent each of the roots graphically, and (c) write each of the roots in standard form.

95. Square roots of $5(\cos 120^\circ + i \sin 120^\circ)$
 96. Square roots of $16(\cos 60^\circ + i \sin 60^\circ)$
 97. Fourth roots of $16\left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}\right)$
 98. Fifth roots of $32\left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}\right)$
 99. Cube roots of $-\frac{125}{2}(1 + \sqrt{3}i)$
 100. Cube roots of $-4\sqrt{2}(1 - i)$

Solving an Equation In Exercises 101–108, use Theorem E.2 to find all the solutions of the equation and represent the solutions graphically.

101. $x^4 - i = 0$ 102. $x^3 + 1 = 0$
 103. $x^5 + 243 = 0$ 104. $x^4 - 81 = 0$
 105. $x^3 + 64i = 0$ 106. $x^6 - 64i = 0$
 107. $x^3 - (1 - i) = 0$ 108. $x^4 + (1 + i) = 0$

109. Proof Given two complex numbers

$$z_1 = r_1(\cos \theta_1 + i \sin \theta_1) \quad \text{and} \quad z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$$

show that

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)], \quad z_2 \neq 0.$$