AP® Exam Practice Questions for Chapter 10

1. $y(t) = 20t - \frac{5}{2}t^2 \qquad x(t) = 2t^3 - 12t^2$

$y'(t) = 20 - 5t \qquad x'(t) = 6t^2 - 24t$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{20 - 5t}{6t^2 - 24t}$$

$$0 = \frac{20 - 5t}{6t^2 - 24t}$$

$$0 = 20 - 5t$$

$$5t = 20$$

$$t = 4$$

$$x(4) = 2(4)^3 - 12(4)^2 = -64$$

$$y(4) = 20(4) - \frac{5}{2}(4)^2 = 40$$

The particle is at rest when $t = 4$ at the point $(-64, 40)$.

So, the answer is B.

2. $A = 2 \cdot \frac{1}{2}\int_0^{\pi} \left(\sin^2 \theta\right)^2 \, d\theta = \int_0^{\pi} \sin^4 \theta \, d\theta$

So, the answer is C.

3. $x = r \cos \theta = \theta \cos \theta \qquad y = r \sin \theta = \theta \sin \theta$

$$\frac{dx}{d\theta} = -\theta \sin \theta + \cos \theta \qquad \frac{dy}{d\theta} = \theta \cos \theta + \sin \theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\theta \cos \theta + \sin \theta}{-\theta \sin \theta + \cos \theta}$$

At $\theta = -\frac{\pi}{2}$,

$$\frac{dy}{dx} = \frac{-\frac{\pi}{2}\cos\left(-\frac{\pi}{2}\right) + \sin\left(-\frac{\pi}{2}\right)}{\frac{\pi}{2}\sin\left(-\frac{\pi}{2}\right) + \cos\left(-\frac{\pi}{2}\right)} = \frac{-1}{-\frac{\pi}{2}} = \frac{2}{\pi}$$

So, the answer is B.

4. $x = 2t^2 + 3t \qquad y = t^3 + 4t^2$

$$\frac{dx}{dt} = 4t + 3 \qquad \frac{dy}{dt} = 3t^2 + 8t$$

At $t = 2$,

$$\frac{dy}{dx} = \frac{dx/dt}{dy/dt} = \frac{3t^2 + 8t}{4t + 3} = \frac{3(2)^2 + 8(2)}{4(2) + 3} = \frac{28}{11}.$$

At $t = 2$, $x(2) = 2(2)^2 + 3(2) = 14$ and

$y(2) = (2)^3 + 4(2)^2 = 24.$

An equation of the tangent line is

$$y - 24 = \frac{28}{11}(x - 14).$$

So, the answer is B.

5. $x = r \cos \theta$

$$= (3 + 5 \sin \theta)(\cos \theta) = 3 \cos \theta + 5 \sin \theta \cos \theta$$

$$\frac{dx}{d\theta} = -3 \sin \theta + 5 \sin \theta(-\sin \theta) + 5(\cos \theta)(\cos \theta) = -3 \sin \theta - 5 \sin^2 \theta + 5 \cos^2 \theta$$

$y = r \sin \theta$

$$= (3 + 5 \sin \theta)(\sin \theta) = 3 \sin \theta + 5 \sin^2 \theta$$

$$\frac{dy}{d\theta} = 3 \cos \theta + 10 \sin \theta \cos \theta$$

At $\theta = 0$,

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3 \cos \theta + 10 \sin \theta \cos \theta}{-3 \sin \theta - 5 \sin^2 \theta + 5 \cos^2 \theta} = \frac{3(1) + 0}{0 - 0 + 5(1)} = \frac{3}{5}.$$

So, the answer is B.

6. $$\begin{aligned} A &= 4\int_0^{\pi/4}\left[(5)^2 - (5\cos 2\theta)^2\right]d\theta \\ &= 4\int_0^{\pi/4}\left(25 - 25\cos^2 2\theta\right)d\theta \\ &= 100\int_0^{\pi/4}\left[1 - \frac{1}{2}\cos^2 2\theta(2)\right]d\theta \\ &= 100\left[\theta - \frac{1}{4}(2\theta + \sin 2\theta\cos 2\theta)\right]_0^{\pi/4} \\ &= 100\left(\left[\frac{\pi}{4} - \frac{1}{4}\left(\frac{\pi}{2} + 0\right)\right] - \left[0 - \frac{1}{4}(0)\right]\right) \\ &= \frac{25\pi}{2} \\ &\approx 39.270 \end{aligned}$$

So, the answer is B.

7. At $t = 4$,

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy/dt}{dx/dt} \\ &= \frac{y'(t)}{x'(t)} \\ &= \frac{2e^{-4t} + 3}{t\cos t} \\ &= \frac{2e^{-4(4)} + 3}{4\cos 4} \\ &\approx -1.147. \end{aligned}$$

So, the answer is B.

8. (a) At $t = 3$, the speed is

$$\begin{aligned} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} &= \sqrt{\left(e^{-t^2+1} - 2\right)^2 + \left(3\sqrt{25 - t^2}\right)^2} \\ &= \sqrt{\left(e^{-3^2+1} - 2\right)^2 + \left(3\sqrt{25 - 3^2}\right)^2} \approx 12.165. \end{aligned}$$

2 pts: 1 pt: expression for speed; 1 pt: answer

(b) $$\begin{aligned} \int_0^5 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt &= \int_0^5 \sqrt{\left(e^{-t^2+1} - 2\right)^2 + \left(3\sqrt{25 - t^2}\right)^2}\,dt \\ &\approx 59.725 \end{aligned}$$

3 pts: 2 pts: integral (arc-length); 1 pt: answer

Note: Be sure to write down the appropriate definite integral before numerically approximating it on your calculator.

(c) $$\begin{aligned} y(3) &= y(0) + \int_0^3 \frac{dy}{dt}\,dt \\ &= 3 + \int_0^3 3\sqrt{25 - t^2}\,dt \\ &\approx 45.131 \end{aligned}$$

4 pts: 2 pts: integral; 1 pt: uses initial condition; 1 pt: answer

Note: Be sure to write down the appropriate definite integral before numerically approximating it on your calculator.

Notes: Round each answer to at least three decimal places to receive credit on the exam.

Use "≈" rather than an equal sign in presenting these approximations from your calculator. Because these are approximations, a point may be deducted if an equal sign is used.

9. (a) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$= \frac{-\sin t}{2t - 5\cos t}$

When $t = 4$, $\frac{dy}{dx} = \frac{-\sin 4}{2(4) - 5\cos 4} = -\frac{\sin 4}{8 - 5\cos 4}$.

So, an equation of the tangent line is

$$y - 3 = -\frac{\sin 4}{8 - 5\cos 4}\left[x - (-1)\right]$$

$$y = 3 - \frac{\sin 4}{8 - 5\cos 4}(x + 1).$$

2 pts: 1 pt: uses $\frac{dy/dt}{dx/dt}$ to compute slope of tangent line; 1 pt: equation of tangent line

(b) $\frac{dx}{dt} = 2t - 5\cos t$

$0 = 2t - 5\cos t$

$t \approx 1.11051$

Because $y'(1.11051) < 0$, the direction is down.

2 pts: 1 pt: t-value with justification $\left(\text{sets } \frac{dx}{dt} = 0\right)$. 1 pt: answer with explanation $\left(\text{considers the sign of } \frac{dy}{dt} \text{ at this } t\text{-value}\right)$

(c) $y(0) = y(4) + \int_4^0 \frac{dy}{dt}\,dt$

$= 3 + \int_4^0 (-\sin t)\,dt$

$= 3 - \int_0^4 (-\sin t)\,dt$

≈ 4.6536

3 pts: 1 pt: definite integral; 1 pt: uses initial condition; 1 pt: answer

Note: Be sure to write down the appropriate definite integral before numerically approximating it on your calculator.

(d) $\int_0^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^4 \sqrt{(-\sin t)^2 + (2t - 5\cos t)^2}\,dt$

≈ 26.657

2 pts: 1 pt: definite integral; 1 pt: answer

Note: Be sure to write down the appropriate definite integral before numerically approximating it on your calculator.

Notes: Be sure to round each answer to at least three decimal places to receive credit on the exam.

Use "≈" rather than an equal sign in presenting these approximations from your calculator. Because these are approximations, a point may be deducted if an equal sign is used.

10. (a) $A = \frac{1}{2}\int_{\pi/2}^{\pi} r^2\, d\theta$

$= \frac{1}{2}\int_{\pi/2}^{\pi} (2\theta + \cos\theta)^2\, d\theta$

≈ 13.338

2 pts: $\begin{cases} \text{1 pt: integral} \\ \text{1 pt: answer} \end{cases}$

Note: Be sure to write down the appropriate definite integral before numerically approximating it on your calculator.

(b) $y = r\sin\theta$

$= (2\theta + \cos\theta)\sin\theta$

$= 2\theta\sin\theta + \sin\theta\cos\theta = 1$

$\theta \approx 2.93575$ radians

$x = r\cos\theta$

$= (2\theta + \cos\theta)\cos\theta$

$= 2\theta\cos\theta + \cos^2\theta$

$x(2.93575) = 2(2.93575)\cos(2.93575) + \cos^2(2.93575)$

≈ -4.790

3 pts: $\begin{cases} \text{1 pt: finds } y(\theta) \\ \text{1 pt: finds } \theta \text{ for which } y = 1 \\ \text{1 pt: finds } x(\theta) \text{ and computes } x(2.93575) \end{cases}$

Notes: Be sure to write down the appropriate equation before numerically approximating its solution on your calculator.

In the intermediate step, round the value of θ to more than three decimal places to use to determine the value of $x(\theta)$.

(c) $x = r\cos\theta$

$= (2\theta + \cos\theta)\cos\theta$

$= 2\theta\cos\theta + \cos^2\theta$

$$\frac{dx}{dt} = 2\theta(-\sin\theta)\frac{d\theta}{dt} + 2\frac{d\theta}{dt}\cdot\cos\theta + 2\cos\theta\cdot(-\sin\theta)\frac{d\theta}{dt}$$

When $\theta = \frac{3\pi}{4}$ and $\frac{d\theta}{dt} = 3$,

$$\frac{dx}{dt} = 2\left(\frac{3\pi}{4}\right)\left(-\sin\frac{3\pi}{4}\right)(3) + 2(3)\left(\cos\frac{3\pi}{4}\right) + 2\left(\cos\frac{3\pi}{4}\right)\left(-\sin\frac{3\pi}{4}\right)(3)$$

$\approx -11.239.$

So, the particle is moving to the left on the rectangular coordinate system.

4 pts: $\begin{cases} \text{2 pts: Chain Rule with respect to } t \\ \quad [\text{computes } (dx/d\theta)\cdot(d\theta/dt)] \\ \text{1 pt: answer} \\ \text{1 pt: interprets answer} \end{cases}$

Notes: Be sure to round each answer to at least three decimal places to receive credit on the exam.

Use "≈" rather than an equal sign in presenting the approximations from your calculator. Because these are approximations, a point may be deducted if an equal sign is used.

11. (a) $A = 3\pi + 2 \cdot \frac{1}{2}\int_{\pi/3}^{\pi} (2 + 2\cos\theta)^2\, d\theta$

≈ 14.197

3 pts: {1 pt: integrand and constant; 1 pt: limits of integration; 1 pt: answer}

Note: Be sure to write down the appropriate definite integral before numerically approximating it on your calculator.

(b) $$\frac{dr}{dt} = \frac{dr}{d\theta}$$

$$\frac{dr}{dt} \cdot \frac{d\theta}{dr} = \frac{dr}{d\theta} \cdot \frac{d\theta}{dr}$$

$$\frac{d\theta}{dt} = 1$$

$$r = 2 + 2\cos\theta$$

$$\frac{dr}{dt} = -2\sin\theta\, \frac{d\theta}{dt} = -2\sin\theta(1) = -2\sin\theta$$

At $\theta = \frac{\pi}{3}$, $\frac{dr}{dt} = -2\sin\frac{\pi}{3} = -2\left(\frac{\sqrt{3}}{2}\right) = -\sqrt{3} \approx -1.732.$

So, the particle is moving closer to the origin of the polar coordinate system.

3 pts: {1 pt: computes $\frac{dr}{dt} = \frac{d\theta}{dt}$; 1 pt: answer; 1 pt: interpretation}

(c) $$\frac{dx}{dt} = \frac{dx}{d\theta}$$

$$\frac{dx}{dt} \cdot \frac{d\theta}{dx} = \frac{dx}{d\theta} \cdot \frac{d\theta}{dx}$$

$$\frac{d\theta}{dt} = 1$$

$$x = r\cos\theta$$

$$\frac{dx}{dt} = r(-\sin\theta)\frac{d\theta}{dt} + \cos\theta\,\frac{dr}{dt}$$

$$= (2 + 2\cos\theta)(-\sin\theta)\frac{d\theta}{dt} + \cos\theta\,\frac{dr}{dt}$$

When $\theta = \frac{\pi}{3}$, $\frac{d\theta}{dt} = 1$, and $\frac{dr}{dt} = -\sqrt{3}$,

$$\frac{dx}{dt} = \left(2 + 2\cos\frac{\pi}{3}\right)\left(-\sin\frac{\pi}{3}\right)(1) + \cos\left(\frac{\pi}{3}\right)\left(-\sqrt{3}\right)$$

$$= (3)\left(-\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{2}\right)\left(-\sqrt{3}\right) = -2\sqrt{3} \approx -3.464$$

So, the particle is moving to the left on the rectangular coordinate system.

3 pts: {1 pt: represents x in terms of θ; 1 pt: computes $\frac{dx}{dt} = \frac{dx}{d\theta}$; 1 pt: answer and interpretation}

Notes: Be sure to round each answer to at least three decimal places to receive credit on the exam.

Use "≈" rather than an equal sign in presenting the approximations from your calculator. Because these are approximations, a point may be deducted if an equal sign is used.

12. (a) $A = \frac{1}{2}\int_0^{\pi/6} (1 - 2\sin\theta)^2\, d\theta$

3 pts: $\begin{cases} \text{1 pt: integrand and constant factor} \\ \text{2 pts: limits of integration} \end{cases}$

(b) $x = r\cos\theta = (1 - 2\sin\theta)\cos\theta = \cos\theta - 2\sin\theta\cos\theta$

$$\frac{dx}{dt} = -\sin\theta\frac{d\theta}{dt} - 2\left[\sin\theta(-\sin\theta) + \cos\theta(\cos\theta)\right]\frac{d\theta}{dt}$$

$$= -\sin\theta\frac{d\theta}{dt} - 2\left(-\sin^2\theta + \cos^2\theta\right)\frac{d\theta}{dt}$$

$$= \left(2\sin^2\theta - 2\cos^2\theta - \sin\theta\right)\frac{d\theta}{dt}$$

$y = r\sin\theta = (1 - 2\sin\theta)(\sin\theta) = \sin\theta - 2\sin^2\theta$

$$\frac{dy}{dt} = (\cos\theta - 4\sin\theta\cos\theta)\frac{d\theta}{dt}$$

3 pts: $\begin{cases} \text{1 pt: writes } x \text{ and } y \text{ as functions of } \theta \\ \text{1 pt: finds } \frac{dx}{dt} = \frac{dx}{d\theta}\cdot\frac{d\theta}{dt} \\ \text{1 pt: finds } \frac{dy}{dt} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dt} \end{cases}$

Note: You do not need to simplify these derivatives.

(c) $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos\theta - 4\sin\theta\cos\theta}{2\sin^2\theta - 2\cos^2\theta - \sin\theta}$

When $\theta = \pi$,

$$\frac{dy}{dx} = \frac{\cos\pi - 4\sin\pi\cos\pi}{2\sin^2\pi - 2\cos^2\pi - \sin\pi}$$

$$= \frac{-1 - 0}{0 - 2(-1)^2 - 0}$$

$$= \frac{1}{2}$$

$x(\theta) = \cos\theta - 2\sin\theta\cos\theta$

$x(\pi) = \cos\pi - 2\sin\pi\cos\pi = -1$

$y(\theta) = \sin\theta - 2\sin^2\theta$

$y(\pi) = \sin\pi = 2\sin^2\pi = 0$

So, an equation of the tangent line is

$$y - 0 = \frac{1}{2}(x + 1)$$

$$y = \frac{1}{2}x + \frac{1}{2}.$$

3 pts: $\begin{cases} \text{1 pt: writes an expression for } dy/dx \text{ in terms of } \theta \\ \text{1 pt: finds values for } x \text{ and } y \text{ when } \theta = \pi \\ \text{1 pt: equation of tangent line} \end{cases}$

13. (a) $\frac{dx}{dt}$ is positive at point *B* because the particle is moving to the right.

$\frac{dy}{dt}$ is positive at point *B* because the particle is moving upward.

2 pts: 1 pt: answer and explanation for the sign of dx/dt; 1 pt: answer and explanation for the sign of dy/dt

(b) Because there is a cusp at point *C*, $\frac{dy}{dt}$ is undefined.

$\frac{dy}{dt}$ is undefined when

$$\left(t^3 - \sqrt{27}\right)^{1/3} = 0$$
$$t^3 = 27^{1/2}$$
$$t = 27^{1/6}.$$

So, the particle reaches point *C* when $t = 27^{1/6} = \sqrt{3}$.

2 pts: 1 pt: reasoning [identifies that dy/dt is undefined at C (cusp)]; 1 pt: answer (finds the t-value where dy/dt is undefined)

(c)
$$y\left(\sqrt{3}\right) = y(0) + \int_0^{\sqrt{3}} y'(t)\,dt$$
$$= 0 - 2\int_0^{\sqrt{3}} \frac{t^2}{\left(t^3 - \sqrt{27}\right)^{1/3}}\,dt \Rightarrow u = t^3 - \sqrt{27},\ du = 3t^2\,dt$$
$$= -2 \cdot \frac{1}{3}\int_0^{\sqrt{3}} \frac{3t^2}{\left(t^3 - \sqrt{27}\right)^{1/3}}\,dt$$
$$= -\frac{2}{3}\left[\frac{3}{2}\left(t^3 - \sqrt{27}\right)^{2/3}\right]_0^{\sqrt{3}}$$
$$= -\left[\left(\left(\sqrt{3}\right)^3 - \sqrt{27}\right)^{2/3} - \left(0 - \sqrt{27}\right)^{2/3}\right]$$
$$= 3$$

3 pts: 1 pt: antiderivative (shows substitution); 1 pt: uses initial condition, $y(0) = 0$; 1 pt: answer

(d)
$$y = -\frac{2}{3}x + \frac{20}{3}$$
$$\frac{dy}{dt} = -\frac{2}{3}\frac{dx}{dt}$$

Because $\frac{dx}{dt} \approx 1.837$, $\frac{dy}{dt} = -\frac{2}{3}(1.837)$.

So, the speed is $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(1.837)^2 + \left[-\frac{2}{3}(1.837)\right]^2}$.

2 pts: 1 pt: finds dy/dt at this point [Chain Rule: $dy/dt = (dy/dx) \cdot (dx/dt)$]; 1 pt: finds the speed

Note: These approximations do not need to be simplified.