AP® Exam Practice Questions for Chapter 9

1. Evaluate each series.

I: $\sum_{n=1}^{\infty} \frac{3}{\sqrt[3]{n}} = 3\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$

Because $p = 1/3$ and $0 < p < 1$, the series diverges.

II: $\sum_{n=1}^{\infty} \left(\frac{e}{\sin 2}\right)^n$

Because $|r| = e/\sin 2 \approx 2.99 > 1$, the series diverges.

III: $\sum_{n=1}^{\infty} \frac{5}{(n+1)!}$

$$\lim_{n\to\infty} \left|\frac{5}{(n+2)!} \cdot \frac{(n+1)!}{5}\right| = \lim_{n\to\infty}\left|\frac{1}{n+2}\right| = 0 < 1$$

By the Ratio Test, the series converges.

So, the answer is C.

2. $\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{4^n}$

$$a_{n+1} = \frac{(n+1)^3}{4^{n+1}} < \frac{n^3}{4^n}$$

$$\lim_{n\to\infty} \frac{n^3}{4^n} = 0$$

By the Alternating Series Test, the series converges.

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(-1)^n n^3}{4^n}\right|} = \lim_{n\to\infty} \frac{(n^3)^{1/n}}{(4^n)^{1/n}} = \lim_{n\to\infty} \frac{n^{3/n}}{4} = \frac{1}{4} < 1$$

By the Root Test, the series converges absolutely.

So, the answer is B.

3. Let $x = \ln 3$.

$$1 + \ln 3 + \frac{(\ln 3)^2}{2!} + \cdots + \frac{(\ln 3)^n}{n!} + \cdots = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = e^x = e^{\ln 3} = 3$$

So, the answer is C.

4. $\sum_{n=2}^{\infty} \frac{5(n+3)}{(n-1)^p}$

Use the Limit Comparison test to compare with $\sum_{n=2}^{\infty} \frac{1}{n^{p-1}}$, which converges for $p - 1 > 1 \Rightarrow p > 2$.

$$\lim_{n\to\infty} \left|\frac{5(n+3)}{(n-1)^p} \cdot \frac{n^{p-1}}{1}\right| = \lim_{n\to\infty}\left|\frac{5n^p + 15n^{p-1}}{(n-1)^p}\right| = 5$$

Because $\lim_{n\to\infty}$ is finite and positive, $\sum_{n=2}^{\infty} \frac{5(n+3)}{(n-1)^p}$ converges when $p > 2$.

So, the answer is C.

5. $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

$$\cos x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!} = 1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \frac{(x^2)^6}{6!} + \frac{(x^2)^8}{8!} - \cdots = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \frac{x^{16}}{8!} - \cdots$$

$$x^2 \cos x^2 = x^2 - \frac{x^2 \cdot x^4}{2!} + \frac{x^2 \cdot x^8}{4!} - \frac{x^2 \cdot x^{12}}{6!} + \frac{x^2 \cdot x^{16}}{8!} - \cdots = x^2 - \frac{x^6}{2!} + \frac{x^{10}}{4!} - \frac{x^{14}}{6!} + \frac{x^{18}}{8!} - \cdots$$

So, the answer is C.

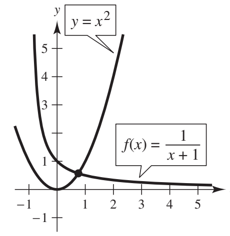
6. $\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{3n^3+1}\right| \Rightarrow a_n = \frac{1}{3n^3+1}$

$$a_{n+1} < 0.001$$
$$\frac{1}{3(n+1)^3+1} < \frac{1}{1000}$$
$$1000 < 3(n+1)^3 + 1$$
$$999 < 3(n+1)^3$$
$$333 < (n+1)^3$$
$$n > -1 + \sqrt[3]{333}$$
$$n > 5.9313$$
$$\Rightarrow n = 6$$

So, the answer is B.

7. $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots$

Graph $f(x) = \frac{1}{1+x}$ and $y = x^2$.



The graphs intersect at $x = 0.755$.

So, the answer is C.

8. (a) $P_1(x) = g(3) + g'(3)(x-3) = 50 + \frac{160}{3}(x-3)$

So, $g(3.1) \approx 50 + \frac{160}{3}(3.1-3) \approx 55.333$.

Because $g'(x)$ is increasing on $[2, 4]$, $g''(3) > 0$. So, $g(x)$ is concave upward at $x = 3$, and the tangent line at $x = 3$ will lie below the graph of g. So, the approximation is less than the actual value of $g(3.1)$.

4 pts: 2 pts: first-degree Taylor polynomial at $x = 3$; 1 pt: approximation of $g(3.1)$; 1 pt: answer (“less than”) with reason

(b) $P_3(x) = g(3) + g'(3)(x-3) + \frac{g''(3)}{2!}(x-3)^2 + \frac{g'''(3)}{3!}(x-3)^3$

$$= 50 + \frac{160}{3}(x-3) + \frac{141}{4\cdot 2!}(x-3)^2 + \frac{21}{3!}(x-3)^3$$
$$= 50 + \frac{160}{3}(x-3) + \frac{141}{8}(x-3)^2 + \frac{7}{2}(x-3)^3$$

So, $g(3.1) \approx 50 + \frac{160}{3}(3.1-3) + \frac{141}{8}(3.1-3)^2 + \frac{7}{2}(3.1-3)^3$

≈ 55.513.

3 pts: 2 pts: third-degree Taylor polynomial at $x = 3$; 1 pt: approximation of $g(3.1)$

(c) $|g(3.1) - P_3(3.1)| = |R_3(3.1)|$

$$= \left|\frac{g^{(4)}(z)}{4!}(x-3)^4\right| \le \frac{1123}{8\cdot 4!}(3.1-3)^4$$
$$= 0.0005849 < 0.0006$$

2 pts: 1 pt: uses the fourth term as an error bound; 1 pt: computes error bound to compare

Notes: Round each answer to at least three decimal places to receive credit for the approximation.

You do not need to simplify the coefficients in these Taylor polynomials.

In these approximations, be sure to write “$g(3.1) \approx$” rather than “$g(3.1) =$.” Because this is an approximation, a point may be deducted if an equal sign is used.

9. (a) $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots + \dfrac{x^n}{n!} + \cdots$

$$e^{-x} = 1 + (-x) + \frac{(-x)^2}{2!} + \cdots + \frac{(-x)^n}{n!} + \cdots$$

$$= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + \frac{(-1)^n x}{n!} + \cdots$$

$$xe^{-x} = x - x \cdot x + \frac{x \cdot x^2}{2!} - \frac{x \cdot x^3}{3!} + \cdots + \frac{(-1)^n \cdot x \cdot x^n}{n!} + \cdots$$

$$= x - x^2 + \frac{x^3}{2!} - \frac{x^4}{3!} + \cdots + \frac{(-1)^n x^{n+1}}{n!} + \cdots$$

So, the first four nonzero terms are $x - x^2 + \dfrac{x^3}{2!} - \dfrac{x^4}{3!}$.

2 pts: {1 pt: first four terms of Taylor series; 1 pt: general term}

(b) $\displaystyle\lim_{x\to 0} \frac{f(x) - x + x^2}{x^3} = \lim_{x\to 0} \frac{xe^{-x} - x + x^2}{x^3} = \frac{1}{2}$

1 pt: answer with justification [using results from part (a)]

(c) $g(x) = \int_0^x te^{-t}\,dt$

$$= \int_0^x \left(t - t^2 + \frac{t^3}{2!} - \frac{t^4}{3!} + \cdots + \frac{(-1)^n t^{n+1}}{n!} + \cdots\right) dt$$

$$= \int_0^x \left(t - t^2 + \frac{t^3}{2} - \frac{t^4}{6} + \cdots + \frac{(-1)^n t^{n+1}}{n!} + \cdots\right) dt$$

$$= \left[\frac{1}{2}t^2 - \frac{1}{3}t^3 + \frac{1}{8}t^4 - \frac{1}{30}t^5 + \cdots + \frac{(-1)^n}{(n+2)n!}t^{n+2} + \cdots\right]_0^x$$

$$= \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{8}x^4 - \frac{1}{30}x^5 + \cdots + \frac{(-1)^n}{(n+2)n!}x^{n+2} + \cdots$$

So, $g\left(\dfrac{1}{5}\right) \approx \dfrac{1}{2}\left(\dfrac{1}{5}\right)^2 - \dfrac{1}{3}\left(\dfrac{1}{5}\right)^3 + \dfrac{1}{8}\left(\dfrac{1}{5}\right)^4$.

4 pts: {2 pts: first four nonzero terms; 1 pt: general term; 1 pt: approximation of $g(1/5)$}

Notes: This approximation does not need to be simplified.

Write "$g(1/5) \approx$" rather than "$g(1/5) =$." Because this is an approximation, a point may be deducted if an equal sign is used.

(d)

$$|a_{n+1}| < \frac{1}{90{,}000}$$

$$\left|\frac{(-1)^{n+1} x^{n+3}}{(n+3)(n+1)!}\right| < \frac{1}{90{,}000}$$

$$\frac{(1/5)^6}{(6)(4!)} < \frac{1}{90{,}000}$$

$$\frac{1}{2{,}250{,}000} < \frac{1}{90{,}000}$$

2 pts: {1 pt: correct form of error bound; 1 pt: computes error bound to compare}

Note: This error bound does not need to be simplified.

10. (a) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$

$$\cos x^2 = 1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \frac{(x^2)^6}{6!} + \cdots + \frac{(-1)^n (x^2)^{2n}}{(2n)!} + \cdots$$

$$= 1 - \frac{x^4}{2} + \frac{x^8}{24} - \frac{x^{12}}{720} + \cdots + \frac{(-1)^n x^{4n}}{(2n)!} + \cdots$$

3 pts: 2 pts: first four terms of series; 1 pt: general term

Note: You do not need to simplify the coefficients in these Taylor polynomials.

(b) $\lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{4(n+1)}}{[2(n+1)]!} \cdot \frac{(2n)!}{(-1)^n x^{4n}} \right| = \lim_{n\to\infty} x^4 \left| \frac{1}{(2n+2)(2n+1)} \right| = 0$

Because the series converges for all x, $R = \infty$.

3 pts: 1 pt: sets up ratio; 1 pt: computes limit of ratio; 1 pt: finds radius of convergence

(c) $\cos x^2 \approx 1 - \frac{x^4}{2} + \frac{x^8}{24}$

$$\cos(1) \approx 1 - \frac{(1)^4}{2} + \frac{(1)^8}{24} = \frac{13}{24}$$

$$|a_{n+1}| = \left| \frac{x^{12}}{6!} \right|$$

$$= \left| \frac{(1)^{12}}{6!} \right|$$

$$= \frac{1}{720} < \frac{1}{500}$$

3 pts: 1 pt: approximation of $f(1)$ using first three terms of series; 1 pt: uses the fourth term as an error bound; 1 pt: computes error bound to compare

Notes: Write "$\cos(1) \approx$" rather than "$\cos(1) =$." Because this is an approximation, a point may be deducted if an equal sign is used.

This error bound does not need to be simplified.

11. (a) $f(x) = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \frac{x^{10}}{5!} + \cdots$

$$f'(x) = -2x + \frac{4x^3}{2} - \frac{6x^5}{6} + \frac{8x^7}{24} - \frac{10x^9}{120} + \cdots$$

$$= -2x + 2x^3 - x^5 + \frac{1}{3}x^7 - \frac{1}{12}x^9 + \cdots$$

$$f''(x) = -2 + 6x^2 - 5x^4 + \frac{7}{3}x^6 - \frac{3}{4}x^8 + \cdots$$

So, $f'(0) = 0$ and $f''(0) = -2$.

Because $f'(0) = 0$, f has a critical value at $x = 0$.

Because $f''(0) < 0$, f is concave downward at $x = 0$.

So, by the Second Derivative Test, f has a relative maximum at $x = 0$.

4 pts:
- 1 pt: finds $f'(0)$
- 1 pt: finds $f''(0)$
- 2 pts: conclusion with reasoning

Note: Explicitly identify each function by name. Referring to "it," "the function," or "the graph" will not receive credit on the exam.

(b) $f(x) = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!}$

$$f(1) = 1 - (1) + \frac{(1)^4}{2!} - \frac{(1)^6}{3!} = \frac{1}{2!} - \frac{1}{3!}$$

Use the Alternating Series Test to find the error.

$$|a_{n+1}| = \left|\frac{x^8}{4!}\right| = \frac{(1)^8}{4!} = \frac{1}{24} < \frac{1}{10}$$

So, $f(1) = \frac{1}{2!} - \frac{1}{3!}$ with an error of less than $\frac{1}{10}$.

3 pts:
- 1 pt: approximation of $f(1)$ using first four terms of series
- 1 pt: uses the fifth term as an error bound
- 1 pt: computes error bound to compare

(c) $y = f(x) = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \frac{x^{10}}{5!} + \cdots$

$$y' = -2x + 2x^3 - x^5 + \frac{1}{3}x^7 - \frac{1}{12}x^9 + \cdots$$

$$y' + 2xy = \left(-2x + 2x^3 - x^5 + \frac{1}{3}x^7 - \frac{1}{12}x^9 + \cdots\right) + 2x\left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \frac{x^{10}}{5!} + \cdots\right)$$

$$= \left(-2x + 2x^3 - x^5 + \frac{1}{3}x^7 - \frac{1}{12}x^9 + \cdots\right) + \left(2x - 2x^3 + x^5 - \frac{1}{3}x^7 + \frac{1}{12}x^9 - \frac{1}{60}x^{11} + \cdots\right)$$

$$= 0$$

2 pts:
- 1 pt: computes $y' = f'(x)$ using given power series
- 1 pt: shows that this differential equation is satisfied

12. (a) $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{(x-1)^{n+1}}{n+1}\cdot\frac{n}{(x-1)^n}\right]$$

$$= \lim_{n\to\infty}\frac{n(x-1)}{n+1}$$

$$= x - 1$$

$$|x-1| < 1$$

$$-1 < x - 1 < 1$$

$$0 < x < 2$$

When $x = 0$: $\frac{(-1)^{n+1}(-1)^n}{n} = \frac{(-1)^{2n+1}}{n} = -\frac{1}{n}$

The series diverges (divergent p-series).

When $x = 2$: $\frac{(-1)^{n+1}(1)^n}{n} = \frac{(-1)^{n+1}}{n}$

The series converges by the Alternating Series Test.

So, the interval of convergence is $(0, 2]$.

4 pts:
- 1 pt: sets up ratio and computes limit of ratio
- 1 pt: finds interior of interval of convergence
- 2 pts: considers both endpoints, shows analysis for interval of convergence

(b)

$$f(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$$

$$g(x) = f'(x) = (1-0) - (x-1) + (x-1)^2 - (x-1)^3 + \cdots$$

$$= 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots + (-1)^n(x-1)^n$$

2 pts:
- 1 pt: first three terms of series
- 1 pt: general term

(c) $g(x) = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots + (-1)^n(x-1)^n$

$$= \frac{1}{x}$$

1 pt: answer

(d) $h(x) = f(x^3 + 1)$

$$h'(x) = f'(x^3+1)\cdot(3x^2)$$

$$= g(x^3+1)(3x^2)$$

$$= \frac{1}{x^3+1}\cdot 3x^2$$

$$= \frac{3x^2}{x^3+1}$$

2 pts: finds $h'(x) = f'(x^3+1) = g(x^3+1)$

[using g from part (c)]

13. (a) Because $g'(2) = 4,$ the equation of the tangent line at $(2, 3)$ is

$$y - 3 = 4(x - 2)$$
$$y = 4x - 5.$$

So, $g(2.4) \approx 4(2.4) - 5 = 4.6.$

Because $g''(2) > 0, g(x)$ is concave upward at $x = 2,$ and the tangent line at $x = 2$ will lie below the graph of g. So, the approximation is less than $g(2.4).$

2 pts: 1 pt: tangent line equation and approximation of $g(2.4)$; 1 pt: answer with reason

(b) $$\int_2^{2.4} g'(x)\,dx = 0.2\left[g'(2.1) + g'(2.3)\right] = 0.2(5 + 8) = 2.6$$

$$g(2.4) \approx g(2) + \int_2^{2.4} g'(x)\,dx = 3 + 2.6 = 5.6$$

2 pts: 1 pt: midpoint Riemann sum approximation; 1 pt: approximation of $g(2.4)$

(c) $$g(2.2) \approx g(2) + 0.2\left[g'(2)\right] = 3 + 0.2(4) = 3.8$$

$$g(2.4) \approx g(2.2) + 0.2\left[g'(2.2)\right] = 3.8 + 0.2(7) = 5.2$$

2 pts: 1 pt: Euler's Method with two steps; 1 pt: answer

(d) $$P_2(x) = g(2) + g'(2)(x - 2) + \frac{g''(2)(x - 2)^2}{2!} = 3 + 4(x - 2) + \frac{20}{2}(x - 2)^2 = 3 + 4(x - 2) + 10(x - 2)^2$$

So, $g(2.4) \approx 3 + 4(2.4 - 2) + 10(2.4 - 2)^2 = 6.2.$

Error: $$\left|\frac{g'''(z)}{3!}(2.4 - 2)^3\right| \le \frac{6}{6}\left(\frac{2}{5}\right)^3 = \frac{8}{125} < \frac{1}{10}$$

3 pts: 1 pt: second-degree Taylor polynomial; 1 pt: approximation of $g(2.4)$; 1 pt: finds error bound using the third term and compares

Notes: These approximations do not need to be simplified.

Write "$g(2.4) \approx$" rather than "$g(2.4) =$." Because these are approximations, a point may be deducted if an equal sign is used.