AP® Exam Practice Questions for Chapter 5

1. $f(x) = 4e^x - x + 6$

$f'(x) = 4e^x - 1 + 0 = 4e^x - 1$

$f'(0) = 4e^0 - 1 = 4(1) - 1 = 3$

Tangent line: $y - 10 = 3(x - 0)$

$y = 3x + 10$

So, the answer is D.

2. $$\frac{dy}{dx} = \frac{d}{dx}\left[6e^x - \frac{\pi \sin x}{4}\right]$$

$$= \frac{d}{dx}(6e^x) - \frac{d}{dx}\left(\frac{\pi}{4}\sin x\right)$$

$$= 6e^x - \frac{\pi}{4}\cos x$$

So, the answer is D.

3. $f(x) = 2x\sqrt{x - 6}$

Because $f(x) = 2x\sqrt{x - 6} = 40$ when $x = 10$, $f(10) = 40$ and $f^{-1}(40) = 10$.

$$\left(f^{-1}\right)'(40) = \frac{1}{f'\left(f^{-1}(40)\right)}$$

$$= \frac{1}{f'(10)}$$

$$f'(x) = 2x\left(\frac{1}{2\sqrt{x - 6}}\right) + 2\sqrt{x - 6}$$

$$= \frac{x}{\sqrt{x - 6}} + 2\sqrt{x - 6}$$

$$= \frac{3x - 12}{\sqrt{x - 6}}$$

$$\left(f^{-1}\right)'(40) = \frac{1}{f'(10)} = \frac{1}{(3(10) - 12)/\sqrt{(10) - 6}}$$

$$= \frac{1}{18/2} = \frac{1}{9}$$

So, the answer is A.

4. $f(x) = \dfrac{1}{3}\arctan\dfrac{x}{3}$

$$f'(x) = \frac{1}{3} \cdot \frac{\frac{1}{3}}{1 + \left(\frac{x}{3}\right)^2} = \frac{1}{9}\left(\frac{1}{1 + \frac{x^2}{9}}\right) = \frac{1}{9 + x^2}$$

So, the answer is D.

5. Because $\displaystyle\lim_{x\to 0}\frac{f(x)}{g(x)} = \frac{0}{0}$, use L'Hôpital's Rule.

$$\lim_{x\to 0}\frac{f'(x)}{g'(x)} = \lim_{x\to 0}\frac{\sec^2 x}{4}$$

$$= \frac{\sec^2 0}{4}$$

$$= \frac{1}{4}$$

So, the answer is B.

6. $$\int \frac{4}{(x - 5)^2 + 9}\,dx$$

Let $u = x - 5$, $du = dx$, and $a = 3$.

$$4\int \frac{1}{u^2 + a^2}\,dx = 4\left[\frac{1}{a}\arctan\frac{u}{a} + C\right]$$

$$= 4\left(\frac{1}{3}\arctan\frac{x - 5}{3} + C\right)$$

$$= \frac{4}{3}\tan^{-1}\frac{x - 5}{3} + C$$

So, the answer is A.

7. $$g'(3) = \frac{1}{f'\left[f^{-1}(3)\right]} = \frac{1}{f'(-2)} = \frac{1}{4}$$

So, the answer is B.

8. $$\frac{d}{dx}\left[\int_0^{x^2} e^{t^2}\,dt\right]$$

Let $u = x^2$ and $du = 2x\,dx$. By the Second Fundamental Theorem of Calculus,

$$\frac{d}{du}\left[\int_0^{u} e^{t^2}\,dt\right]\frac{du}{dx} = e^{u^2}(2x)$$

$$= 2xe^{x^4}.$$

So, the answer is C.

9. $$x \ln y = 2$$

$$x\left(\frac{1}{y}\frac{dy}{dx}\right) + (1)\ln y = 0$$

$$\frac{x}{y}\frac{dy}{dx} = -\ln y$$

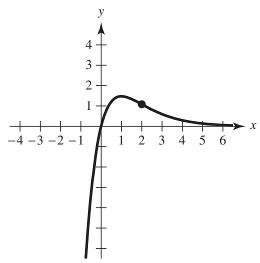
$$\frac{dy}{dx} = -\frac{y \ln y}{x}$$

So, the answer is B.

10. $h(x) = 4xe^{-x}$

$h'(x) = 4e^{-x} - 4xe^{-x}$

$h''(x) = -4e^{-x} - \left(4e^{-x} - 4xe^{-x}\right) = -4e^{-x}(2 - x) = 0$ when $x = 2$.



Because $h'' < 0$ on $(-\infty, 2)$, $h'' > 0$ on $(2, \infty)$, and $x = 2$ is a point of inflection, the graph of $h(x)$ is decreasing and concave upward on $(2, \infty)$.

So, the answer is A.

11. $\lim_{x\to 1} \dfrac{5e^{1-x} - \ln x - 5}{x^2 - 1} = \lim_{x\to 1} \dfrac{-5e^{1-x} - (1/x)}{2x} = \dfrac{-5e^0 - (1/1)}{2(1)} = \dfrac{-5 - 1}{2} = -3$

So, the answer is A.

12. $f(x) = \ln(x - 3)$

$f'(x) = \dfrac{1}{x - 3}$

By the Mean Value Theorem, $f'(c) = \dfrac{f(b) - f(a)}{b - a} = \dfrac{f(8) - f(4)}{8 - 4} = \dfrac{\ln 5 - \ln 1}{4} = \dfrac{\ln 5}{4}$

$$f'(x) = f'(c)$$
$$\frac{1}{x - 3} = \frac{\ln 5}{4}$$
$$x - 3 = \frac{4}{\ln 5}$$
$$x = \frac{4}{\ln 5} + 3$$
$$x \approx 5.485$$

So, the answer is A.

13. $f(x) = \cos^{-1} x$

$f'(x) = -\dfrac{1}{\sqrt{1 - x^2}}$

Tangent line at $\left(\dfrac{1}{2}, \dfrac{\pi}{3}\right)$:

$$y - f\left(\frac{1}{2}\right) = f'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right)$$
$$y - \frac{\pi}{3} = -\frac{1}{\sqrt{1 - \frac{1}{4}}}\left(x - \frac{1}{2}\right)$$
$$y = -\frac{2\sqrt{3}}{3}x + \frac{\sqrt{3} + \pi}{3}$$

At $x = 0.52$, $y = \dfrac{-2\sqrt{3}}{3}(0.52) + \dfrac{\sqrt{3} + \pi}{3} \approx 1.024$.

So, the answer is C.

14. (a) $v(t) = \sin\left(\frac{5\pi}{2}e^{-t/\pi}\right) < 0$ when $0.701 < t < 2.879.$

So, the particle is traveling to the left on the interval $(0.701, 2.879)$.

1 pt: answer (considers where $v(t) < 0$)

(b) Average velocity $= \frac{1}{4}\int_0^4 v(t)\,dt$

$$= \frac{1}{4}\int_0^4 \sin\left(\frac{5\pi}{2}e^{-t/\pi}\right)dt$$

$$\approx -0.103$$

2 pts:
- 1 pt: integral
- 1 pt: answer

(c) $v(t) = \sin\left(\frac{5\pi}{2}e^{-t/\pi}\right)$

$$v(2) = \sin\left(\frac{5\pi}{2}e^{-2/\pi}\right) \approx -0.849$$

$$a(t) = v'(t) = \left[\cos\left(\frac{5\pi}{2}e^{-t/\pi}\right)\right]\left(\frac{5\pi}{2}e^{-t/\pi}\right)\left(-\frac{1}{\pi}\right)$$

$$a(2) \approx 0.699$$

Because $v(2) < 0$ and $a(2) > 0$, the velocity is negative and increasing. So, the particle is slowing down at $t = 2$.

2 pts:
- 1 pt: finds $v(2)$ and $a(2)$
- 1 pt: answer with reason (compares signs of $v(2)$ and $a(2)$)

Note: $a(2) = v'(2)$ can be computed numerically on your calculator (no work needed for this).

(d) Because $v(t) = 0$ when $t \approx 0.701$ and $t \approx 2.879$, these are the critical numbers of $v(t)$.

Let $x(t)$ be the position function for the particle.

$$x(0) = 2$$

$$x(0.701) = 2 + \int_0^{0.701} v(t)dt \approx 2.433$$

$$x(2.879) = 2 + \int_0^{2.879} v(t)dt \approx 1.070$$

$$x(4) = 2 + \int_0^{4} v(t)dt \approx 1.588$$

So, the maximum distance from the origin is 2.433.

4 pts:
- 1 pt: finds $x(0.701)$ using a definite integral
- 1 pt: finds $x(2.879)$ using a definite integral
- 1 pt: finds $x(4)$ using a definite integral
- 1 pt: answer (must consider the value of x at each critical point and each endpoint of the interval to be eligible for the answer point. Note: $x(0) = 2$ is given.)

Notes: Round each answer to at least three decimal places to receive credit on the exam.

Use "≈" rather than an equal sign in presenting these approximations from your calculator. Since these are approximations, a point may be deducted if an equal sign is used.

Write down the approximate definite integrals—including the differential, *dt*—before numerically approximating these integrals on your calculator. (No additional work is needed.)

15. (a) $f(x) = \dfrac{e^x + e^{-x}}{2}$

$$f'(x) = \frac{1}{2}\left(e^x - e^{-x}\right)$$

$$0 = \frac{1}{2}\left(e^x - e^{-x}\right)$$

$$e^x = e^{-x}$$

$$x = 0$$

$f'(x) < 0$ when $x < 0$ and $f'(x) > 0$ when $x > 0$.

So, there is a relative minimum at $x = 0$.

3 pts:
- 1 pt: considers where $f'(x) = 0$ to locate the critical number
- 2 pts: reason (explains that $f'(x)$ changes from negative to positive at $x = 0$)

Note: In such an explanation, reason using $f'(x)$ (explain using a derivative). Merely reasoning with f (appealing to where f changes from decreasing to increasing) may not receive credit on the exam.

(b) Average value $= \dfrac{1}{2}\displaystyle\int_{-1}^{1} \frac{e^x + e^{-x}}{2}\, dx$

≈ 1.175

3 pts:
- 2 pts: definite integral
- 1 pt: answer

Notes: Be sure to write down the appropriate definite integral before numerically approximating it on your calculator.

Be sure to round the answer to at least three decimal places to receive credit on the exam.

(c) $f(x) = \dfrac{1}{2}\left(e^x + e^{-x}\right)$

$$f'(x) = \frac{1}{2}\left(e^x - e^{-x}\right)$$

$$f''(x) = \frac{1}{2}\left(e^x + e^{-x}\right)$$

Because $e^x > 0$ and $e^{-x} > 0$ for all values of x, $f''(x) > 0$ for all values of x.

So, f is concave upward on $(-\infty, \infty)$ and does not have any inflection points.

3 pts:
- 1 pt: find $f''(x)$
- 2 pts: answer with reason (establishes there are no x-values for which $f''(x)$ equals 0 or is undefined, or reasons with the sign of $f''(x)$)

16. (a) $g(0) = f(0)$

$$= \frac{1}{\sqrt{4 - 0^2}}$$

$$= \frac{1}{2}$$

$$\lim_{x \to 0^-} g(x) = \frac{1}{2}$$

$$\lim_{x \to 0^+} g(x) = 0 + \frac{1}{2} = \frac{1}{2}$$

Because $\lim_{x \to 0^-} g(x) = \lim_{x \to 0^+} g(x) = g(0)$, by the definition of continuity, g is continuous at $x = 0$.

4 pts:
- 1 pt: finds $g(0)$ using the appropriate function
- 2 pts: finds each one-sided limit to find $\lim_{x \to 0} g(x)$
- 1 pt: shows $g(0) = \lim_{x \to 0} g(x)$ to reach conclusion (uses the definition of continuous)

(b) $A = \int_0^1 f(x)\, dx$

≈ 0.524

2 pts:
- 1 pt: integral
- 1 pt: answer

Notes: Be sure to write down the appropriate definite integral before numerically approximating this integral on your calculator.

Be sure to round the answer to at least three decimal places to receive credit on the exam.

(c) $\int_{-1}^1 g(x)\, dx = \int_{-1}^0 f(x)\, dx + \int_0^1 \left(x + \frac{1}{2}\right) dx$

$$= \int_{-1}^0 \frac{1}{\sqrt{4 - x^2}}\, dx + \int_0^1 \left(x + \frac{1}{2}\right) dx$$

≈ 1.524

3 pts:
- 2 pts: splits the given integral into two appropriate integrals (1 point for each definite integral)
- 1 pt: answer

Notes: Be sure to write down the appropriate definite integrals before numerically approximating these integrals on your calculator.

Be sure to round the answer to at least three decimal places to receive credit on the exam.

17. (a) $\int_0^2 f(t)\,dt = \int_0^2 \left(-2\sqrt{t} + 5\right) dt$

≈ 6.229

So, the amount of water entering the basement is about 6.229 cubic feet.

2 pts: {1 pt: integral; 1 pt: answer}

Note: Be sure to write down the appropriate definite integral before numerically approximating this integral on your calculator.

(b) $f(2) - g(2) = \left(-2\sqrt{2} + 5\right) - 5\left(1 - e^{-0.5(2)}\right)$

$= -2\sqrt{2} + \dfrac{5}{e}$

≈ -0.989

At $t = 2$, the water level is changing at a rate of about -0.989 cubic feet per hour.

2 pts: {1 pt: considers $f(2) - g(2)$; 1 pt: answer with units}

(c) $R(t) = f(t) - g(t)$

$= \left(-2\sqrt{t} + 5\right) - 5\left(1 - e^{-0.5t}\right)$

$= -2\sqrt{t} + 5e^{-0.5t}$

$R(t) = 0$ when $t \approx 1.457$.

Because $R(t) > 0$ on $(0, 1.457)$ and $R(t) < 0$ on $(1.457, 5)$, the volume of water is at a maximum at $t \approx 1.457$.

The total amount of water at $t \approx 1.457$ is

$15 + \int_0^{1.457} R(t)\,dt \approx 17.829$ cubic feet.

5 pts:
- 1 pt: finds critical number (determines where $R(t) = f(t) - g(t) = 0$)
- 1 pt: answer with reason (explains that $R(t)$ changes from positive to negative at this critical number)
- 2 pts: sets up definite integral and uses initial condition to represent this maximum volume of water
- 1 pt: answer with units

Note: Be sure to write down the appropriate definite integral before numerically approximating this integral on your calculator.

Note: Be sure to round each answer to at least three decimal places to receive credit on the exam.

18. (a) Because the towers are 366 feet apart, the towers are located at $x = \frac{366}{2} = 183$ and $x = -183$.

$$f(183) = \frac{125}{2}\left(e^{183/250} + e^{-183/250}\right)$$
$$\approx 160$$

So, the towers are about 160 feet tall.

1 pt: answer with units

(b) $$f(x) = \frac{125}{2}\left(e^{x/250} + e^{-x/250}\right)$$
$$f'(x) = \frac{125}{2}e^{x/250}\left(\frac{1}{250}\right) + \frac{125}{2}e^{-x/250}\left(-\frac{1}{250}\right)$$
$$= \frac{1}{4}\left(e^{x/250} - e^{-x/250}\right)$$

So, $f'(100) = \frac{1}{4}\left(e^{100/250} - e^{-100/250}\right) \approx 0.205$, which represents the slope of the cable at a point 83 feet from the building on the right.

2 pts: $\begin{cases}\text{1 pt: answer}\\ \text{1 pt: explanation}\end{cases}$

Note: $f'(100)$ can be computed numerically on your calculator (no work is needed).

(c) $$\text{Average height} = \frac{1}{366}\int_{-183}^{183} f(x)\,dx$$
$$= \frac{125}{732}\int_{-183}^{183}\left(e^{x/250} + e^{-x/250}\right)dx$$
$$\approx 136.466 \text{ ft}$$

3 pts: $\begin{cases}\text{2 pts: integral}\\ \text{1 pt: answer}\end{cases}$

Note: Be sure to write down the appropriate definite integral before numerically approximating this integral on your calculator.

(d) Because the average height is 136.466 feet, $f(x) = 136.466$ when $x \approx 106.27734$ feet.

$$f'(x) = \frac{1}{4}\left(e^{x/250} - e^{-x/250}\right)$$
$$f'(106.27734) = \frac{1}{4}\left(e^{106.27734/250} - e^{-106.27734/250}\right)$$
$$\approx 0.219$$

So, the slope of the cable is about 0.219.

3 pts: $\begin{cases}\text{1 pt: finds where } f(x) = \text{ average height}\\ \text{2 pts: evaluates } f'(x) \text{ at this } x\text{-value}\end{cases}$

Note: Be sure to round each answer to at least three decimal places to receive credit on the exam.

19. (a) $h(x) = f(g(x))$

$= f(x^2)$

$= \arccos x^2$

$h'(x) = \dfrac{-2x}{\sqrt{1 - x^4}} = 0$ when $-2x = 0 \Rightarrow x = 0$.

Because $h'(x)$ changes sign from positive to negative at $x = 0$, $h(x)$ has a relative maximum at $x = 0$.

4 pts:
- 2 pts: finds $h'(x)$ (applies the Chain Rule)
- 2 pts: answer with explanation [appeals to $h'(x)$ changing from positive to negative at $x = 0$]

Note: In such an explanation, reason using $h'(x)$ (explain using a derivative). Merely reasoning with *h* (appealing to where *h* changes from increasing to decreasing) may not receive credit on the exam.

(b) Find the points of intersection of $h(x)$ and *y*.

$h(x) = y$

$\arccos x^2 = \dfrac{\pi}{3}$

$x^2 = \cos\dfrac{\pi}{3}$

$x^2 = \dfrac{1}{2}$

$x = \pm\dfrac{\sqrt{2}}{2}$

So, the area can be determined by

$A = \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \left(\arccos x^2 - \dfrac{\pi}{3}\right) dx.$

3 pts:
- 2 pts: limits of integration (finds the *x*-values of the intersection points)
- 1 pt: integrand (must have a definite integral to be eligible for this point)

Note: If writing this as a single integral, be sure to use parentheses to correctly represent this integrand. Missing parentheses or, equivalently, and incorrectly placed *dx* may result in the loss of this point on the exam.

(c) $f(x) = \arccos x$

$f'(x) = -\dfrac{1}{\sqrt{1 - x^2}}$

$\left(f^{-1}\right)'\left(\dfrac{\pi}{3}\right) = \dfrac{1}{f'\left[f^{-1}\left(\dfrac{\pi}{3}\right)\right]}$

$= \dfrac{1}{f'\left(\cos\dfrac{\pi}{3}\right)}$

$= \dfrac{1}{f'\left(\dfrac{1}{2}\right)}$

$= \dfrac{1}{-1/\sqrt{1 - (1/2)^2}}$

$= -\sqrt{\dfrac{3}{4}}$

$= -\dfrac{\sqrt{3}}{2}$

2 pts:
- 1 pt: represents $\left(f^{-1}\right)'\left(\dfrac{\pi}{3}\right)$ as $\dfrac{1}{f'\left(\cos\dfrac{\pi}{3}\right)}$
- 1 pt: answer

20. (a) $f(x) = e^x - x$

$f'(x) = e^x - 1$

$0 = e^x - 1$

$1 = e^x$

$x = 0$

Because $f'(x) < 0$ for $x < 0$ and $f'(x) > 0$ for $x > 0$, $x = 0$ is a relative minimum.

3 pts: {1 pt: finds critical number; 2 pts: classifies and explains ($f'(x)$ changes from negative to positive at $x = 0$)}

Note: In such an explanation, reason using $f'(x)$ (explain using a derivative). Merely reasoning with f (appealing to where f changes from decreasing to increasing) may not receive credit on the exam.

(b) $f'(x) = e^x - 1$ $\quad$ $f(x) = e^x - x$

$f'(1) = e^1 - 1$ $\quad$ $f(1) = e^1 - 1 = e - 1$

$= e - 1$

So, an equation of the tangent line is

$y - (e - 1) = (e - 1)(x - 1)$

$y = (e - 1)x - (e - 1) + (e - 1)$

$y = (e - 1)x.$

3 pts: {2 pts: finds the slope of this tangent line (computes $f'(1)$); 1 pt: writes equation of tangent line}

(c) $\int_0^a f(x)\,dx = \int_0^a (e^x - x)\,dx$

$= \left[e^x - \frac{1}{2}x^2\right]_0^a$

$= \left(e^a - \frac{1}{2}a^2\right) - e^0$

$= e^a - \frac{1}{2}a^2 - 1$

Because $f'(x) = e^x - 1$, $f'(a) = e^a - 1$.

$\int_0^a f(x)\,dx = f'(a)$

$e^a - \frac{1}{2}a^2 - 1 = e^a - 1$

$-\frac{1}{2}a^2 = 0$

$a = 0$

3 pts: {2 pts: evaluates $\int_0^a f(x)\,dx$; 1 pt: answer (solves equation)}

21. (a) $f'(1.5) = \dfrac{f(2) - f(1)}{2 - 1}$

$= \dfrac{-1 - 3}{1}$

$= -4$

2 pts:
- 1 pt: justification (evidence of a difference quotient on $[1, 2]$)
- 1 pt: answer

(b) Because f is continuous and differentiable, by the Mean Value Theorem, there must exist some value c, in $(1, 3)$, such that

$$f'(c) = \frac{f(3) - f(1)}{3 - 1} = \frac{5 - 3}{2} = 1.$$

3 pts:
- 1 pt: finds the average rate of change over $[1, 3]$ (evidence of a difference quotient on $[1, 3]$).
- 1 pt: establishes that f is continuous and differentiable on $[1, 3]$ (or everywhere)
- 1 pt: appeals to the Mean Value Theorem to justify the existence of such a c.

(c) Because $f(3) = 5$, $f^{-1}(5) = 3$.

$$\left(f^{-1}\right)'(5) = \frac{1}{f'\left[f^{-1}(5)\right]}$$

$$= \frac{1}{f'(3)}$$

$$= \frac{1}{-2}$$

So, an equation of the tangent line is

$$y - 3 = -\frac{1}{2}(x - 5)$$

$$y = -\frac{1}{2}x + \frac{5}{2} + 3$$

$$y = -\frac{1}{2}x + \frac{11}{2}.$$

4 pts:
- 1 pt: finds/states $f^{-1}(5) = 3$
- 1 pt: represents $\left(f^{-1}\right)'(5)$ as $\dfrac{1}{f'\left[f^{-1}(5)\right]}$
- 1 pt: finds $\left(f^{-1}\right)'(5)$
- 1 pt: finds the equation of tangent line