16 Additional Topics in Differential Equations

Parachute Jump
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Electrical Circuits (Exercises 29 and 30, p. 1151)

Undamped or Damped Motion? (Exercise 53, p. 1144)

Motion of a Spring
(Example 8, p. 1142)

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Clockwise from top left, Phovoir/Shutterstock.com; Danshutter/Shutterstock.com;
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16.1 Exact First-Order Equations

- **Solve an exact differential equation.**
- **Use an integrating factor to make a differential equation exact.**

Exact Differential Equations

In Chapter 6, you studied applications of differential equations to growth and decay problems. You also learned more about the basic ideas of differential equations and studied the solution technique known as separation of variables. In this chapter, you will learn more about solving differential equations and using them in real-life applications. This section introduces you to a method for solving the first-order differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

for the special case in which this equation represents the exact differential of a function $z = f(x, y)$.

Definition of an Exact Differential Equation

The equation

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is an **exact differential equation** when there exists a function f of two variables x and y having continuous partial derivatives such that

$$f_x(x, y) = M(x, y) \quad \text{and} \quad f_y(x, y) = N(x, y).$$

The general solution of the equation is $f(x, y) = C$.

From Section 13.3, you know that if f has continuous second partials, then

$$\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial N}{\partial x}.$$

This suggests the following test for exactness.

THEOREM 16.1 Test for Exactness

Let M and N have continuous partial derivatives on an open disk R. The differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Every differential equation of the form

$$M(x)\,dx + N(y)\,dy = 0$$

is exact. In other words, a separable differential equation is actually a special type of an exact equation.

Exactness is a fragile condition in the sense that seemingly minor alterations in an exact equation can destroy its exactness. This is demonstrated in the next example.

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EXAMPLE 1 Testing for Exactness

Determine whether each differential equation is exact.

a. $(xy^2 + x)\,dx + x^2y\,dy = 0$ **b.** $\cos y\,dx + (y^2 - x\sin y)\,dy = 0$

Solution

a. This differential equation is exact because

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}[xy^2 + x] = 2xy \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}[x^2y] = 2xy.$$

Notice that the equation $(y^2 + 1)\,dx + xy\,dy = 0$ is not exact, even though it is obtained by dividing each side of the first equation by x.

b. This differential equation is exact because

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}[\cos y] = -\sin y \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}[y^2 - x\sin y] = -\sin y.$$

Notice that the equation $\cos y\,dx + (y^2 + x\sin y)\,dy = 0$ is not exact, even though it differs from the first equation only by a single sign.

Note that the test for exactness of $M(x, y)\,dx + N(x, y)\,dy = 0$ is the same as the test for determining whether $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ is the gradient of a potential function (Theorem 15.1). This means that a general solution $f(x, y) = C$ to an exact differential equation can be found by the method used to find a potential function for a conservative vector field.

EXAMPLE 2 Solving an Exact Differential Equation

See LarsonCalculus.com for an interactive version of this type of example.

Solve the differential equation $(2xy - 3x^2)\,dx + (x^2 - 2y)\,dy = 0$

Solution This differential equation is exact because

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}[2xy - 3x^2] = 2x \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}[x^2 - 2y] = 2x.$$

The general solution, $f(x, y) = C$, is

$$f(x, y) = \int M(x, y)\,dx = \int (2xy - 3x^2)\,dx = x^2y - x^3 + g(y).$$

In Section 15.1, you determined $g(y)$ by integrating $N(x, y)$ with respect to y and reconciling the two expressions for $f(x, y)$. An alternative method is to partially differentiate this version of $f(x, y)$ with respect to y and compare the result with $N(x, y)$. In other words,

$$f_y(x, y) = \frac{\partial}{\partial y}[x^2y - x^3 + g(y)] = x^2 + g'(y) = \overbrace{x^2 - 2y}^{N(x,\,y)}.$$

$g'(y) = -2y$

So, $g'(y) = -2y$, and it follows that $g(y) = -y^2 + C_1$. Therefore,

$$f(x, y) = x^2y - x^3 - y^2 + C_1$$

and the general solution is $x^2y - x^3 - y^2 = C$.

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EXAMPLE 3 Solving an Exact Differential Equation

Find the particular solution of $(\cos x - x \sin x + y^2)\,dx + 2xy\,dy = 0$ that satisfies the initial condition $y = 1$ when $x = \pi$.

Solution The differential equation is exact because

$$\overbrace{\frac{\partial}{\partial y}[\cos x - x \sin x + y^2]}^{\frac{\partial M}{\partial y}} = 2y = \overbrace{\frac{\partial}{\partial x}[2xy]}^{\frac{\partial N}{\partial x}}.$$

Because $N(x, y)$ is simpler than $M(x, y)$, it is better to begin by integrating $N(x, y)$.

$$f(x, y) = \int N(x, y)\,dy = \int 2xy\,dy = xy^2 + g(x)$$

Next, find $f_x(x, y)$ and compare the result with $M(x, y)$.

$$f_x(x, y) = \frac{\partial}{\partial x}[xy^2 + g(x)] = y^2 + g'(x) = \overbrace{\cos x - x \sin x + y^2}^{M(x,\,y)}$$

$g'(x) = \cos x - x \sin x$

So, $g'(x) = \cos x - x \sin x$ and it follows that

$$g(x) = \int (\cos x - x \sin x)\,dx$$
$$= x \cos x + C_1.$$

This implies that $f(x, y) = xy^2 + x \cos x + C_1$, and the general solution is

$$xy^2 + x \cos x = C. \qquad \text{General solution}$$

Applying the given initial condition produces

$$\pi(1)^2 + \pi \cos \pi = C$$

which implies that $C = 0$. So, the particular solution is

$$xy^2 + x \cos x = 0.$$

The graph of the particular solution is shown in Figure 16.2. Notice that the graph consists of two parts: the ovals are given by $y^2 + \cos x = 0$, and the y-axis is given by $x = 0$.

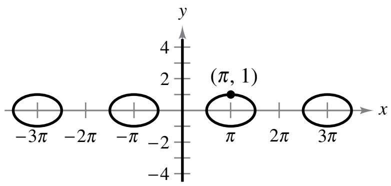

Figure 16.2

TECHNOLOGY A graphing utility can be used to graph a particular solution that satisfies the initial condition of a differential equation. In Example 3, the differential equation and initial condition are satisfied when $xy^2 + x \cos x = 0$, which implies that the particular solution can be written as $x = 0$ or $y = \pm\sqrt{-\cos x}$. On a graphing utility screen, the solution would be represented by Figure 16.1 together with the y-axis.

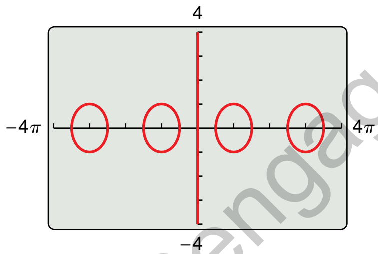

Figure 16.1

In Example 3, note that for $z = f(x, y) = xy^2 + x \cos x$, the total differential of z is given by

$$dz = f_x(x, y)\,dx + f_y(x, y)\,dy$$
$$= (\cos x - x \sin x + y^2)\,dx + 2xy\,dy$$
$$= M(x, y)\,dx + N(x, y)\,dy.$$

In other words, $M\,dx + N\,dy = 0$ is called an *exact* differential equation because $M\,dx + N\,dy$ is exactly the differential of $f(x, y)$.

Integrating Factors

When the differential equation $M(x, y)\,dx + N(x, y)\,dy = 0$ is not exact, it may be possible to make it exact by multiplying by an appropriate factor $u(x, y)$, which is called an **integrating factor** for the differential equation.

EXAMPLE 4 Multiplying by an Integrating Factor

a. When the differential equation

$$2y\,dx + x\,dy = 0 \qquad \text{Not an exact equation}$$

is multiplied by the integrating factor $u(x, y) = x$, the resulting equation

$$2xy\,dx + x^2\,dy = 0 \qquad \text{Exact equation}$$

is exact—the left side is the total differential of x^2y.

b. When the equation

$$y\,dx - x\,dy = 0 \qquad \text{Not an exact equation}$$

is multiplied by the integrating factor $u(x, y) = 1/y^2$, the resulting equation

$$\frac{1}{y}\,dx - \frac{x}{y^2}\,dy = 0 \qquad \text{Exact equation}$$

is exact—the left side is the total differential of x/y.

Finding an integrating factor can be difficult. There are two classes of differential equations, however, whose integrating factors can be found routinely—namely, those that possess integrating factors that are functions of either x alone or y alone. The next theorem, which is presented without proof, outlines a procedure for finding these two special categories of integrating factors.

REMARK When either $h(x)$ or $k(y)$ is constant, Theorem 16.2 still applies. As an aid to remembering these formulas, note that the subtracted partial derivative identifies both the denominator and the variable for the integrating factor.

THEOREM 16.2 Integrating Factors

Consider the differential equation $M(x, y)\,dx + N(x, y)\,dy = 0$.

1. If

$$\frac{1}{N(x, y)}[M_y(x, y) - N_x(x, y)] = h(x)$$

is a function of x alone, then $e^{\int h(x)\,dx}$ is an integrating factor.

2. If

$$\frac{1}{M(x, y)}[N_x(x, y) - M_y(x, y)] = k(y)$$

is a function of y alone, then $e^{\int k(y)\,dy}$ is an integrating factor.

Exploration

In Chapter 6, you solved the first-order linear differential equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

by using the integrating factor $u(x) = e^{\int P(x)\,dx}$. Show that you can obtain this integrating factor by using the methods of this section.

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EXAMPLE 5 Finding an Integrating Factor

Solve the differential equation $(y^2 - x)\,dx + 2y\,dy = 0$.

Solution This equation is not exact because

$$M_y(x, y) = 2y \quad \text{and} \quad N_x(x, y) = 0.$$

However, because

$$\frac{M_y(x, y) - N_x(x, y)}{N(x, y)} = \frac{2y - 0}{2y} = 1 = h(x)$$

it follows that $e^{\int h(x)\,dx} = e^{\int dx} = e^x$ is an integrating factor. Multiplying the differential equation by e^x produces the exact differential equation

$$(y^2e^x - xe^x)\,dx + 2ye^x\,dy = 0.$$

Next, integrate $N(x, y)$, as shown.

$$f(x, y) = \int N(x, y)\,dy = \int 2ye^x\,dy = y^2e^x + g(x)$$

Now, find $f_x(x, y)$ and compare the result with $M(x, y)$.

$$f_x(x, y) = y^2e^x + g'(x) = \overbrace{y^2e^x - xe^x}^{M(x,\,y)}$$

$g'(x) = -xe^x$

Therefore, $g'(x) = -xe^x$ and $g(x) = -xe^x + e^x + C_1$, which implies that

$$f(x, y) = y^2e^x - xe^x + e^x + C_1.$$

The general solution is $y^2e^x - xe^x + e^x = C$, or

$$y^2 - x + 1 = Ce^{-x}. \qquad \text{General solution}$$

The next example shows how a differential equation can help in sketching a force field given by $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$.

EXAMPLE 6 An Application to Force Fields

Sketch the force field

$$\mathbf{F}(x, y) = \frac{2y}{\sqrt{x^2 + y^2}}\mathbf{i} - \frac{y^2 - x}{\sqrt{x^2 + y^2}}\mathbf{j}$$

by finding and sketching the family of curves tangent to $\mathbf{F}$.

Solution At the point (x, y) in the plane, the vector $\mathbf{F}(x, y)$ has a slope of

$$\frac{dy}{dx} = \frac{-(y^2 - x)/\sqrt{x^2 + y^2}}{2y/\sqrt{x^2 + y^2}} = \frac{-(y^2 - x)}{2y}$$

which, in differential form, is

$$2y\,dy = -(y^2 - x)\,dx$$
$$(y^2 - x)\,dx + 2y\,dy = 0.$$

From Example 5, you know that the general solution of this differential equation is $y^2 = x - 1 + Ce^{-x}$. Figure 16.3 shows several representative curves from this family. Note that the force vector at (x, y) is tangent to the curve passing through (x, y).

Force field:
$\mathbf{F}(x, y) = \frac{2y}{\sqrt{x^2 + y^2}}\mathbf{i} - \frac{y^2 - x}{\sqrt{x^2 + y^2}}\mathbf{j}$
Family of curves tangent to $\mathbf{F}$:
$y^2 = x - 1 + Ce^{-x}$

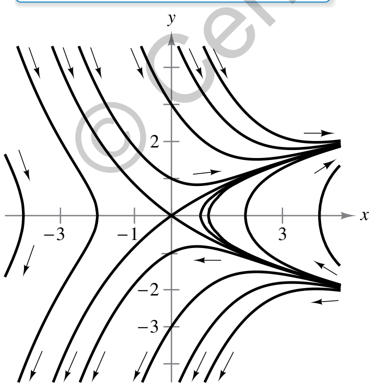

Figure 16.3

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16.1 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. Exactness What does it mean for the differential equation $M(x, y)\,dx + N(x, y)\,dy = 0$ to be exact? Explain how to determine whether this differential equation is exact.

2. Integrating Factor When is it beneficial to use an integrating factor to find the solution of the differential equation $M(x, y)\,dx + N(x, y)\,dy = 0$?

Testing for Exactness **In Exercises 3–6, determine whether the differential equation is exact.**

3. $(2x + xy^2)\,dx + (3 + x^2y)\,dy = 0$

4. $(2xy - y)\,dx + (x^2 - xy)\,dy = 0$

5. $x \sin y\,dx + x \cos y\,dy = 0$

6. $ye^{xy}\,dx + xe^{xy}\,dy = 0$

Solving an Exact Differential Equation **In Exercises 7–14, verify that the differential equation is exact. Then find the general solution.**

7. $(2x - 3y)\,dx + (2y - 3x)\,dy = 0$

8. $ye^x\,dx + e^x\,dy = 0$

9. $(3y^2 + 10xy^2)\,dx + (6xy - 2 + 10x^2y)\,dy = 0$

10. $2\cos(2x - y)\,dx - \cos(2x - y)\,dy = 0$

11. $\dfrac{1}{x^2 + y^2}(x\,dy - y\,dx) = 0$

12. $e^{-(x^2+y^2)}(x\,dx + y\,dy) = 0$

13. $\dfrac{x}{y^2}\,dx - \dfrac{x^2}{y^3}\,dy = 0$

14. $(e^y \cos xy)[y\,dx + (x + \tan xy)\,dy] = 0$

Graphical and Analytic Analysis **In Exercises 15 and 16, (a) sketch an approximate solution of the differential equation satisfying the initial condition on the slope field, (b) find the particular solution that satisfies the initial condition, and (c) use a graphing utility to graph the particular solution. Compare the graph with the sketch in part (a).**

	Differential Equation	Initial Condition
15.	$(2x \tan y + 5)\,dx + (x^2 \sec^2 y)\,dy = 0$	$y\left(\dfrac{1}{2}\right) = \dfrac{\pi}{4}$
16.	$\dfrac{1}{\sqrt{x^2 + y^2}}(x\,dx + y\,dy) = 0$	$y(4) = 3$

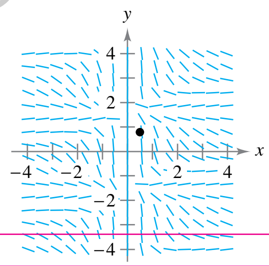


Figure for 15

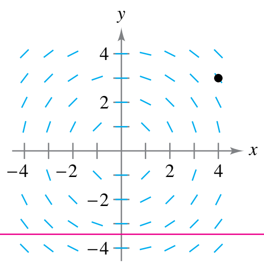


Figure for 16

Finding a Particular Solution **In Exercises 17–22, find the particular solution of the differential equation that satisfies the initial condition.**

17. $(2xy - 9x^2)\,dx + (2y + x^2 + 1)\,dy = 0, \quad y(0) = -3$

18. $(2xy^2 + 4)\,dx + (2x^2y - 6)\,dy = 0, \quad y(-1) = 8$

19. $e^{3x}(\sin 3y\,dx + \cos 3y\,dy) = 0, \quad y(0) = \pi$

20. $(x^2 + y^2)\,dx + 2xy\,dy = 0, \quad y(3) = 1$

21. $\dfrac{y}{x - 1}\,dx + [\ln(x - 1) + 2y]\,dy = 0, \quad y(2) = 4$

22. $\dfrac{1}{x^2 + y^2}(x\,dx + y\,dy) = 0, \quad y(0) = 4$

Finding an Integrating Factor **In Exercises 23–32, find the integrating factor that is a function of x or y alone and use it to find the general solution of the differential equation.**

23. $y^2\,dx + 5xy\,dy = 0$

24. $(2x^3 + y)\,dx - x\,dy = 0$

25. $y\,dx - (x + 6y^2)\,dy = 0$

26. $(5x^2 - y^2)\,dx + 2y\,dy = 0$

27. $(x + y)\,dx + \tan x\,dy = 0$

28. $(2x^2y - 1)\,dx + x^3\,dy = 0$

29. $y^2\,dx + (xy - 1)\,dy = 0$

30. $(x^2 + 2x + y)\,dx + 2\,dy = 0$

31. $2y\,dx + \left(x - \sin\sqrt{y}\right)\,dy = 0$

32. $(-2y^3 + 1)\,dx + (3xy^2 + x^3)\,dy = 0$

Using an Integrating Factor **In Exercises 33–36, use the integrating factor to find the general solution of the differential equation.**

	Integrating Factor	Differential Equation
33.	$u(x, y) = xy^2$	$(4x^2y + 2y^2)\,dx + (3x^3 + 4xy)\,dy = 0$
34.	$u(x, y) = x^2y$	$(3y^2 + 5x^2y)\,dx + (3xy + 2x^3)\,dy = 0$
35.	$u(x, y) = x^{-2}y^{-3}$	$(-y^5 + x^2y)\,dx + (2xy^4 - 2x^3)\,dy = 0$
36.	$u(x, y) = x^{-2}y^{-2}$	$-y^3\,dx + (xy^2 - x^2)\,dy = 0$

37. Integrating Factor Show that each expression is an integrating factor for the differential equation $y\,dx - x\,dy = 0$.

(a) $\dfrac{1}{x^2}$ (b) $\dfrac{1}{y^2}$ (c) $\dfrac{1}{xy}$ (d) $\dfrac{1}{x^2 + y^2}$

38. Integrating Factor Show that the differential equation $(axy^2 + by)\,dx + (bx^2y + ax)\,dy = 0$ is exact only when $a = b$. For $a \neq b$, show that x^my^n is an integrating factor, where

$$m = -\frac{2b + a}{a + b}, \quad n = -\frac{2a + b}{a + b}.$$

Tangent Curves **In Exercises 39–42, use a graphing utility to graph the family of curves tangent to the force field.**

39. $\mathbf{F}(x, y) = \dfrac{y}{\sqrt{x^2 + y^2}}\mathbf{i} - \dfrac{x}{\sqrt{x^2 + y^2}}\mathbf{j}$

40. $\mathbf{F}(x, y) = \dfrac{x}{\sqrt{x^2 + y^2}}\mathbf{i} - \dfrac{y}{\sqrt{x^2 + y^2}}\mathbf{j}$

41. $\mathbf{F}(x, y) = 4x^2y\mathbf{i} - \left(2xy^2 + \dfrac{x}{y^2}\right)\mathbf{j}$

42. $\mathbf{F}(x, y) = (1 + x^2)\mathbf{i} - 2xy\mathbf{j}$

Finding an Equation of a Curve **In Exercises 43 and 44, find an equation of the curve with the specified slope passing through the given point.**

	Slope	Point
43.	$\dfrac{dy}{dx} = \dfrac{y - x}{3y - x}$	$(2, 1)$
44.	$\dfrac{dy}{dx} = \dfrac{-2xy}{x^2 + y^2}$	$(-1, 1)$

45. Cost

In a manufacturing process where $y = C(x)$ represents the cost of producing x units, the **elasticity of cost** is defined as

$$E(x) = \frac{\text{marginal cost}}{\text{average cost}} = \frac{C'(x)}{C(x)/x} = \frac{x}{y}\frac{dy}{dx}.$$

Find the cost function when the elasticity function is

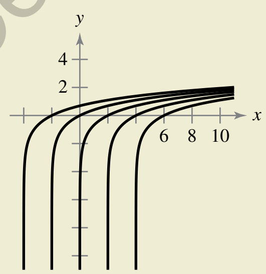
$$E(x) = \frac{20x - y}{2y - 10x}$$

where

$$C(100) = 500$$

and $x \geq 100$.

46. **HOW DO YOU SEE IT?** The graph shows several representative curves from the family of curves tangent to a force field **F**. Which is the equation of the force field? Explain your reasoning.



(a) $\mathbf{F}(x, y) = -\mathbf{i} + 2\mathbf{j}$ (b) $\mathbf{F}(x, y) = -3x\mathbf{i} + y\mathbf{j}$
(c) $\mathbf{F}(x, y) = e^x\mathbf{i} - \mathbf{j}$ (d) $\mathbf{F}(x, y) = 2\mathbf{i} + e^{-y}\mathbf{j}$

Euler's Method **In Exercises 47 and 48, (a) use Euler's Method and a graphing utility to graph the particular solution of the differential equation over the indicated interval with the specified value of h and initial condition, (b) find the particular solution of the differential equation analytically, and (c) use a graphing utility to graph the particular solution and compare the result with the graph in part (a).**

	Differential Equation	Interval	h	Initial Condition
47.	$y' = \dfrac{-xy}{x^2 + y^2}$	$[2, 4]$	0.05	$y(2) = 1$
48.	$y' = \dfrac{6x + y^2}{y(3y - 2x)}$	$[0, 5]$	0.2	$y(0) = 1$

49. **Euler's Method** Repeat Exercise 47 for $h = 1$ and discuss how the accuracy of the result changes.

50. **Euler's Method** Repeat Exercise 48 for $h = 0.5$ and discuss how the accuracy of the result changes.

EXPLORING CONCEPTS

Exact Differential Equation **In Exercises 51 and 52, find all values of k such that the differential equation is exact.**

51. $(xy^2 + kx^2y + x^3)\,dx + (x^3 + x^2y + y^2)\,dy = 0$

52. $(ye^{2xy} + 2x)\,dx + (kxe^{2xy} - 2y)\,dy = 0$

53. **Exact Differential Equation** Find all nonzero functions f and g such that

$$g(y)\sin x\,dx + y^2 f(x)\,dy = 0$$

is exact.

54. **Exact Differential Equation** Find all nonzero functions g such that

$$g(y)e^y\,dx + xy\,dy = 0$$

is exact.

True or False? **In Exercises 55–58, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

55. Every separable equation is an exact equation.

56. Every exact equation is a separable equation.

57. If $M\,dx + N\,dy = 0$ is exact, then

$$[f(x) + M]\,dx + [g(y) + N]\,dy = 0$$

is also exact.

58. If $M\,dx + N\,dy = 0$ is exact, then

$$xM\,dx + xN\,dy = 0$$

is also exact.

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16.2 Second-Order Homogeneous Linear Equations

- **Solve a second-order linear differential equation.**
- **Solve a higher-order linear differential equation.**
- **Use a second-order linear differential equation to solve an applied problem.**

Second-Order Linear Differential Equations

In this section and the next section, you will learn methods for solving higher-order linear differential equations.

Definition of Linear Differential Equation of Order *n*

Let $g_1, g_2, . . . , g_n$ and f be functions of x with a common (interval) domain. An equation of the form

$$y^{(n)} + g_1(x)y^{(n-1)} + g_2(x)y^{(n-2)} + \cdot \cdot \cdot + g_{n-1}(x)y' + g_n(x)y = f(x)$$

is a **linear differential equation of order *n*.** If $f(x) = 0$, then the equation is **homogeneous;** otherwise, it is **nonhomogeneous.**

REMARK Notice that this use of the term *homogeneous* differs from that in Section 6.3.

Homogeneous equations are discussed in this section, and the nonhomogeneous case is discussed in the next section.

The functions $y_1, y_2, . . . , y_n$ are **linearly independent** when the *only* solution of the equation

$$C_1y_1 + C_2y_2 + \cdot \cdot \cdot + C_ny_n = 0$$

is the trivial one, $C_1 = C_2 = \cdot \cdot \cdot = C_n = 0$. Otherwise, this set of functions is **linearly dependent.**

EXAMPLE 1 Linearly Independent and Dependent Functions

Determine whether the functions are linearly independent or linearly dependent.

a. $y_1(x) = \sin x,\ y_2(x) = x$ **b.** $y_1(x) = x,\ y_2(x) = 3x$

Solution

a. The functions $y_1(x) = \sin x$ and $y_2(x) = x$ are linearly independent because the only values of C_1 and C_2 for which

$$C_1 \sin x + C_2 x = 0$$

for all x are $C_1 = 0$ and $C_2 = 0$.

b. It can be shown that two functions form a linearly dependent set if and only if one is a constant multiple of the other. For example, $y_1(x) = x$ and $y_2(x) = 3x$ are linearly dependent because

$$C_1x + C_2(3x) = 0$$

has the nonzero solutions $C_1 = -3$ and $C_2 = 1$.

The theorem on the next page points out the importance of linear independence in constructing the general solution of a second-order linear homogeneous differential equation with constant coefficients.

THEOREM 16.3 Linear Combinations of Solutions

If y_1 and y_2 are linearly independent solutions of the differential equation $y'' + ay' + by = 0$, then the general solution is

$$y = C_1y_1 + C_2y_2 \qquad \text{General solution}$$

where C_1 and C_2 are constants.

Proof Letting y_1 and y_2 be solutions of $y'' + ay' + by = 0$, you obtain the following system of equations.

$$y_1''(x) + ay_1'(x) + by_1(x) = 0$$
$$y_2''(x) + ay_2'(x) + by_2(x) = 0$$

Multiplying the first equation by C_1, multiplying the second by C_2, and adding the resulting equations together, you obtain

$$[C_1y_1''(x) + C_2y_2''(x)] + a[C_1y_1'(x) + C_2y_2'(x)] + b[C_1y_1(x) + C_2y_2(x)] = 0$$

which means that $y = C_1y_1 + C_2y_2$ is a solution, as desired. The proof that all solutions are of this form is best left to a full course on differential equations. ∎

Theorem 16.3 states that when you can find two linearly independent solutions, you can obtain the general solution by forming a **linear combination** of the two solutions.

To find two linearly independent solutions, note that the nature of the equation $y'' + ay' + by = 0$ suggests that it may have solutions of the form $y = e^{mx}$. If so, then

$$y' = me^{mx} \quad \text{and} \quad y'' = m^2e^{mx}.$$

So, by substitution, $y = e^{mx}$ is a solution if and only if

$$y'' + ay' + by = 0$$
$$m^2e^{mx} + ame^{mx} + be^{mx} = 0$$
$$e^{mx}(m^2 + am + b) = 0.$$

Because e^{mx} is never 0, $y = e^{mx}$ is a solution if and only if

$$m^2 + am + b = 0. \qquad \text{Characteristic equation}$$

This is the **characteristic equation** of the differential equation $y'' + ay' + by = 0$. Note that the characteristic equation can be determined from its differential equation simply by replacing y'' with m^2, y' with m, and y with 1.

EXAMPLE 2 Characteristic Equation: Distinct Real Zeros

Solve the differential equation $y'' - 4y = 0$.

Solution In this case, the characteristic equation is

$$m^2 - 4 = 0. \qquad \text{Characteristic equation}$$

So, $m = \pm 2$. Therefore, $y_1 = e^{m_1x} = e^{2x}$ and $y_2 = e^{m_2x} = e^{-2x}$ are particular solutions of the differential equation. Furthermore, because these two solutions are linearly independent, you can apply Theorem 16.3 to conclude that the general solution is

$$y = C_1e^{2x} + C_2e^{-2x}. \qquad \text{General solution}$$ ∎

Exploration

For each differential equation below, find the characteristic equation. Solve the characteristic equation for m, and use the values of m to find a general solution of the differential equation. Using your results, develop a general solution of differential equations with characteristic equations that have distinct real roots.

(a) $y'' - 9y = 0$

(b) $y'' - 6y' + 8y = 0$

The characteristic equation in Example 2 has two distinct real zeros. From algebra, you know that this is only one of *three* possibilities for quadratic equations. In general, the quadratic equation $m^2 + am + b = 0$ has zeros

$$m_1 = \frac{-a + \sqrt{a^2 - 4b}}{2} \quad \text{and} \quad m_2 = \frac{-a - \sqrt{a^2 - 4b}}{2}$$

which fall into one of three cases.

1. Two distinct real zeros, $m_1 \neq m_2$
2. Two equal real zeros, $m_1 = m_2$
3. Two complex conjugate zeros, $m_1 = \alpha + \beta i$ and $m_2 = \alpha - \beta i$

In terms of the differential equation $y'' + ay' + by = 0$, these three cases correspond to three different types of general solutions.

THEOREM 16.4 Solutions of $y'' + ay' + by = 0$

The solutions of $y'' + ay' + by = 0$ fall into one of three cases, depending on the solutions of the characteristic equation, $m^2 + am + b = 0$.

1. *Distinct Real Zeros* If $m_1 \neq m_2$ are distinct real zeros of the characteristic equation, then the general solution is

$$y = C_1e^{m_1x} + C_2e^{m_2x}.$$

2. *Equal Real Zeros* If $m_1 = m_2$ are equal real zeros of the characteristic equation, then the general solution is

$$y = C_1e^{m_1x} + C_2xe^{m_1x} = (C_1 + C_2x)e^{m_1x}.$$

3. *Complex Zeros* If $m_1 = \alpha + \beta i$ and $m_2 = \alpha - \beta i$ are complex zeros of the characteristic equation, then the general solution is

$$y = C_1e^{\alpha x}\cos \beta x + C_2e^{\alpha x}\sin \beta x.$$

FOR FURTHER INFORMATION For more information on Theorem 16.4, see the article "A Note on a Differential Equation" by Russell Euler in the 1989 winter issue of the *Missouri Journal of Mathematical Sciences.*

EXAMPLE 3 Characteristic Equation: Complex Zeros

Find the general solution of the differential equation $y'' + 6y' + 12y = 0$.

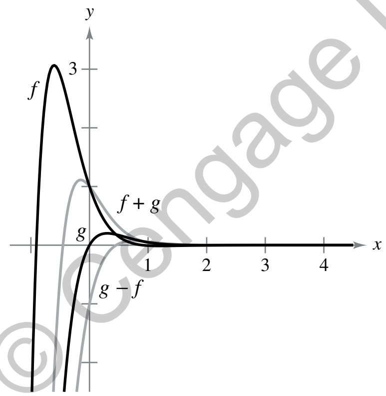
Solution The characteristic equation $m^2 + 6m + 12 = 0$ has two complex zeros, as follows.

$$\begin{aligned} m &= \frac{-6 \pm \sqrt{36 - 48}}{2} && \text{Use Quadratic Formula with } a = 1,\ b = 6, \text{ and } c = 12. \\ &= \frac{-6 \pm \sqrt{-12}}{2} \\ &= \frac{-6 \pm 2\sqrt{-3}}{2} \\ &= -3 \pm \sqrt{-3} \\ &= -3 \pm \sqrt{3}i \end{aligned}$$

So, $\alpha = -3$ and $\beta = \sqrt{3}$, and the general solution is

$$y = C_1e^{-3x}\cos \sqrt{3}x + C_2e^{-3x}\sin \sqrt{3}x.$$

Several members of the family of solutions, including $f(x) = e^{-3x}\cos \sqrt{3}x$ and $g(x) = e^{-3x}\sin \sqrt{3}x$, are shown in Figure 16.4. (Note that although the characteristic equation has two *complex* zeros, the solution of the differential equation is *real*.)



Several members of the family of solutions to Example 3, including $f(x) = e^{-3x}\cos \sqrt{3}x$ and $g(x) = e^{-3x}\sin \sqrt{3}x$, are shown in the graph. Notice that as $x \rightarrow \infty$, all of these solutions approach 0.

Figure 16.4

EXAMPLE 4 **Characteristic Equation: Repeated Zeros**

Solve the differential equation

$$y'' + 4y' + 4y = 0$$

subject to the initial conditions $y(0) = 2$ and $y'(0) = 1$.

Solution The characteristic equation and its zeros are

$$m^2 + 4m + 4 = 0$$
$$(m + 2)^2 = 0$$
$$m = -2.$$

So, the characteristic equation has two equal zeros given by $m = -2$, and the general solution is

$$y = C_1 e^{-2x} + C_2 x e^{-2x}.$$ General solution

Now, because $y = 2$ when $x = 0$, you have

$$2 = C_1(1) + C_2(0)(1)$$
$$2 = C_1.$$

Furthermore, because $y' = 1$ when $x = 0$, you have

$$y' = -2C_1 e^{-2x} + C_2(-2xe^{-2x} + e^{-2x})$$ Derivative of y with respect to x
$$1 = -2(2)(1) + C_2[-2(0)(1) + 1]$$ Substitute.
$$1 = -4 + C_2$$
$$5 = C_2.$$

Therefore, the particular solution is

$$y = 2e^{-2x} + 5xe^{-2x}.$$ Particular solution

Try checking this solution in the original differential equation.

Higher-Order Linear Differential Equations

For higher-order homogeneous linear differential equations, you can find the general solution in much the same way as you do for second-order equations. That is, you begin by determining the n zeros of the characteristic equation. Then, based on these n zeros, you form a linearly independent collection of n solutions. The major difference is that with equations of third or higher order, zeros of the characteristic equation may occur more than twice. When this happens, the linearly independent solutions are formed by multiplying by increasing powers of x, as demonstrated in Examples 6 and 7.

EXAMPLE 5 **Solving a Third-Order Equation**

Find the general solution of

$$y''' - y' = 0.$$

Solution The characteristic equation and its zeros are

$$m^3 - m = 0$$
$$m(m - 1)(m + 1) = 0$$
$$m = 0, 1, -1.$$

Because the characteristic equation has three distinct zeros, the general solution is

$$y = C_1 + C_2 e^{-x} + C_3 e^x.$$ General solution

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EXAMPLE 6 Solving a Third-Order Equation

Find the general solution of

$$y''' + 3y'' + 3y' + y = 0.$$

Solution The characteristic equation and its zeros are

$$m^3 + 3m^2 + 3m + 1 = 0$$
$$(m + 1)^3 = 0$$
$$m = -1.$$

Because the zero $m = -1$ occurs three times, the general solution is

$$y = C_1e^{-x} + C_2xe^{-x} + C_3x^2e^{-x}.$$

General solution

EXAMPLE 7 Solving a Fourth-Order Equation

See LarsonCalculus.com for an interactive version of this type of example.

Find the general solution of

$$y^{(4)} + 2y'' + y = 0.$$

Solution The characteristic equation and its zeros are

$$m^4 + 2m^2 + 1 = 0$$
$$(m^2 + 1)^2 = 0$$
$$m = \pm i.$$

Because each of the zeros

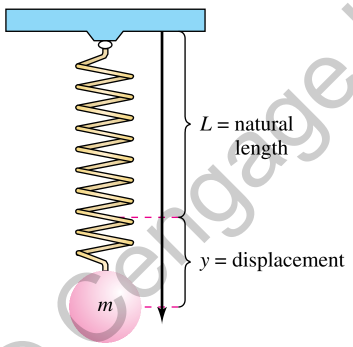
$$m_1 = \alpha + \beta i = 0 + i \quad \text{and} \quad m_2 = \alpha - \beta i = 0 - i$$

occurs twice, the general solution is

$$y = C_1 \cos x + C_2 \sin x + C_3 x \cos x + C_4 x \sin x.$$

General solution

Application



A rigid object of mass m attached to the end of the spring causes a displacement of y.
Figure 16.5

One of the many applications of linear differential equations is describing the motion of an oscillating spring. According to Hooke's Law, a spring that is stretched (or compressed) y units from its natural length L tends to *restore* itself to its natural length by a force F that is proportional to y. That is, $F(y) = -ky$, where k is the **spring constant** and indicates the stiffness of the spring.

A rigid object of mass m is attached to the end of a spring and causes a displacement, as shown in Figure 16.5. Assume that the mass of the spring is negligible compared with m. When the object is pulled downward and released, the resulting oscillations are a product of two opposing forces—the spring force $F(y) = -ky$ and the weight mg of the object. Under such conditions, you can use a differential equation to find the position y of the object as a function of time t. According to Newton's Second Law of Motion, the force acting on the weight is $F = ma$, where $a = d^2y/dt^2$ is the acceleration. Assuming that the motion is **undamped**—that is, there are no other external forces acting on the object—it follows that

$$m\frac{d^2y}{dt^2} = -ky$$

and you have

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0.$$

Undamped motion of a spring

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EXAMPLE 8 Undamped Motion of a Spring

A 4-pound weight stretches a spring 8 inches from its natural length. The weight is pulled downward an additional 6 inches and released with an initial upward velocity of 8 feet per second. Find a formula for the position of the weight as a function of time t.

Solution The 4-pound weight stretches the spring 8 inches $= \frac{2}{3}$ foot from its natural length, so by Hooke's Law

$$4 = k\left(\frac{2}{3}\right) \quad \Longrightarrow \quad k = 6.$$

Moreover, because the weight w is given by mg, it follows that

$$m = \frac{w}{g} = \frac{4}{32} = \frac{1}{8}.$$

So, the resulting differential equation for this undamped motion is

$$\frac{d^2y}{dt^2} + \left(\frac{6}{1/8}\right)y = 0 \quad \Longrightarrow \quad \frac{d^2y}{dt^2} + 48y = 0.$$

The characteristic equation $m^2 + 48 = 0$ has complex zeros $m = 0 \pm 4\sqrt{3}i$, so the general solution is

$$\begin{aligned} y &= C_1e^0 \cos 4\sqrt{3}t + C_2e^0 \sin 4\sqrt{3}t \\ &= C_1 \cos 4\sqrt{3}t + C_2 \sin 4\sqrt{3}t. \end{aligned}$$

When $t = 0$ seconds, $y = 6$ inches $= \frac{1}{2}$ foot. Using this initial condition, you have

$$\frac{1}{2} = C_1(1) + C_2(0) \quad \Longrightarrow \quad C_1 = \frac{1}{2}. \qquad y(0) = \tfrac{1}{2}$$

To determine C_2, note that $y' = 8$ feet per second when $t = 0$ seconds.

$$y' = -4\sqrt{3}C_1 \sin 4\sqrt{3}t + 4\sqrt{3}C_2 \cos 4\sqrt{3}t \qquad \text{Derivative of } y \text{ with respect to } t$$

$$8 = -4\sqrt{3}\left(\frac{1}{2}\right)(0) + 4\sqrt{3}C_2(1) \qquad \text{Substitute.}$$

$$\frac{2\sqrt{3}}{3} = C_2$$

Consequently, the position at time t is given by

$$y = \frac{1}{2}\cos 4\sqrt{3}t + \frac{2\sqrt{3}}{3}\sin 4\sqrt{3}t.$$

A common type of spring is a *coil spring,* also called a *helical spring* because the shape of the spring is a helix. A *tension* coil spring resists being stretched (see photo above and Example 8). A *compression* coil spring resists being compressed, such as the spring in a car suspension.

The object in Figure 16.6 undergoes an additional damping or frictional force that is proportional to its velocity. A case in point would be the damping force resulting from friction and movement through a fluid. Considering this damping force

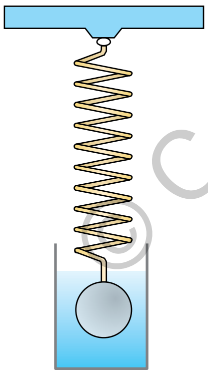
$$-p\frac{dy}{dt} \qquad \text{Damping force}$$

the differential equation for the oscillation is

$$m\frac{d^2y}{dt^2} = -ky - p\frac{dy}{dt}$$

or, in standard linear form,

$$\frac{d^2y}{dt^2} + \frac{p}{m}\left(\frac{dy}{dt}\right) + \frac{k}{m}y = 0. \qquad \text{Damped motion of a spring}$$



A damped vibration could be caused by friction and movement through a liquid.
Figure 16.6

16.2 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Linear Differential Equation** Determine the order of each linear differential equation and decide whether each equation is homogeneous.
   (a) $y^{(5)} + x^6y' + xy = 0$ (b) $y'' + 3e^xy + 2x = 0$
2. **Linearly Independent** Describe what it means for the functions y_1 and y_2 to be linearly independent.
3. **Using Zeros** The zeros of the characteristic equation for two differential equations of the form $y'' + ay' + by = 0$ are given. Write the corresponding general solution for each set of zeros.
   (a) $m = -1, 3$ (b) $m = 2, 2$
4. **Finding a General Solution** Explain how to find the general solution of a higher-order homogenous linear differential equation.

Verifying a Solution In Exercises 5–8, verify the solution of the differential equation. Then use a graphing utility to graph the particular solutions for several different values of C_1 and C_2.

Solution	Differential Equation
5. $y = (C_1 + C_2x)e^{-3x}$	$y'' + 6y' + 9y = 0$
6. $y = C_1 + C_2e^{3x}$	$y'' - 3y' = 0$
7. $y = C_1 \cos 2x + C_2 \sin 2x$	$y'' + 4y = 0$
8. $y = C_1e^{-x} \cos 3x + C_2e^{-x} \sin 3x$	$y'' + 2y' + 10y = 0$

Finding a General Solution In Exercises 9–36, find the general solution of the linear differential equation.

9. $y'' - y' = 0$
10. $y'' + 2y' = 0$
11. $y'' - y' - 6y = 0$
12. $y'' + 6y' + 5y = 0$
13. $2y'' + 3y' - 2y = 0$
14. $16y'' - 16y' + 3y = 0$
15. $y'' + 6y' + 9y = 0$
16. $y'' - 10y' + 25y = 0$
17. $16y'' - 8y' + y = 0$
18. $9y'' - 12y' + 4y = 0$
19. $y'' + y = 0$
20. $y'' + 4y = 0$
21. $4y'' - 5y = 0$
22. $y'' - 2y = 0$
23. $y'' - 2y' + 4y = 0$
24. $y'' - 4y' + 21y = 0$
25. $y'' - 3y' + y = 0$
26. $3y'' + 4y' - y = 0$
27. $9y'' - 12y' + 11y = 0$
28. $2y'' - 6y' + 7y = 0$
29. $y^{(4)} - y = 0$
30. $y^{(4)} - y'' = 0$
31. $y''' - 6y'' + 11y' - 6y = 0$
32. $y''' - y'' - y' + y = 0$
33. $y''' - 3y'' + 7y' - 5y = 0$
34. $y''' - 3y'' + 3y' - y = 0$
35. $y^{(4)} - 2y'' + y = 0$
36. $y^{(4)} - 2y''' + y'' = 0$

37. **Finding a Particular Solution** Consider the differential equation $y'' + 100y = 0$ and the solution $y = C_1 \cos 10x + C_2 \sin 10x$. Find the particular solution satisfying each initial condition.
    (a) $y(0) = 2,\ y'(0) = 0$ (b) $y(0) = 0,\ y'(0) = 2$
    (c) $y(0) = -1,\ y'(0) = 3$
38. **Finding a Particular Solution** Determine C and ω such that $y = C \sin \sqrt{3}t$ is a particular solution of the differential equation $y'' + \omega y = 0$, where $y'(0) = -5$.

Finding a Particular Solution: Initial Conditions In Exercises 39–44, find the particular solution of the linear differential equation that satisfies the initial conditions.

39. $y'' - y' - 30y = 0$
    $y(0) = 1,\ y'(0) = -4$
40. $y'' - 7y' + 12y = 0$
    $y(0) = 3,\ y'(0) = 3$
41. $y'' + 16y = 0$
    $y(0) = 0,\ y'(0) = 2$
42. $9y'' - 6y' + y = 0$
    $y(0) = 2,\ y'(0) = 1$
43. $y'' + 2y' + 3y = 0$
    $y(0) = 2,\ y'(0) = 1$
44. $4y'' + 4y' + y = 0$
    $y(0) = 3,\ y'(0) = -1$

Finding a Particular Solution: Boundary Conditions In Exercises 45–50, find the particular solution of the linear differential equation that satisfies the boundary conditions.

45. $y'' - 4y' + 3y = 0$
    $y(0) = 1,\ y(1) = 3$
46. $4y'' + y = 0$
    $y(0) = 2,\ y(\pi) = -5$
47. $y'' + 9y = 0$
    $y(0) = 3,\ y\left(\frac{\pi}{2}\right) = 4$
48. $4y'' + 20y' + 21y = 0$
    $y(0) = 3,\ y(2) = 0$
49. $4y'' - 28y' + 49y = 0$
    $y(0) = 2,\ y(1) = -1$
50. $y'' + 6y' + 45y = 0$
    $y(0) = 4,\ y\left(\frac{\pi}{12}\right) = 2$

EXPLORING CONCEPTS

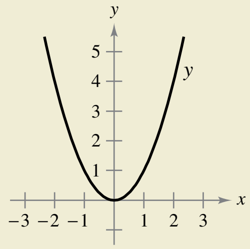
51. **Finding Another Solution** Show that the equation $y = C_1 \sinh x + C_2 \cosh x$ is a solution of the homogeneous linear differential equation $y'' - y = 0$. Then use hyperbolic definitions to find another solution of the differential equation.
52. **General Solution of a Differential Equation** What is the general solution of $y^{(n)} = 0$? Explain.

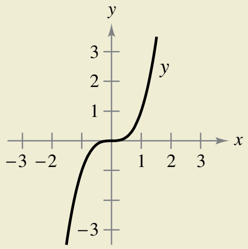
53. Undamped or Damped Motion?

Several shock absorbers are shown at the right. Do you think the motion of the spring in a shock absorber is undamped or damped?

54. **HOW DO YOU SEE IT?** Give a geometric argument to explain why the graph cannot be a solution of the differential equation. (It is not necessary to solve the differential equation.)

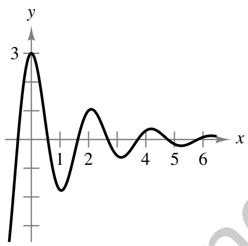
(a) $y'' = y'$ (b) $y'' = -\frac{1}{2}y'$



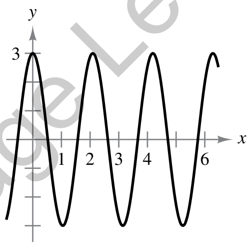


Motion of a Spring **In Exercises 55–58, match the differential equation with the graph of a particular solution. [The graphs are labeled (a), (b), (c), and (d).] The correct match can be made by comparing the frequency of the oscillations or the rate at which the oscillations are being damped with the appropriate coefficient in the differential equation.**

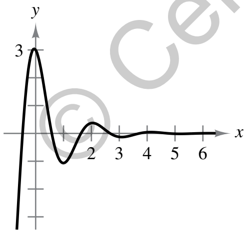
(a)



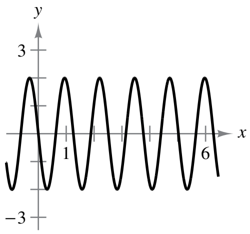
(b)



(c)



(d)



55. $y'' + 9y = 0$

56. $y'' + 25y = 0$

57. $y'' + 2y' + 10y = 0$

58. $y'' + y' + \frac{37}{4}y = 0$

Motion of a Spring **In Exercises 59–64, a 32-pound weight stretches a spring $\frac{2}{3}$ foot from its natural length. Use the given information to find a formula for the position of the weight as a function of time.**

59. The weight is pulled $\frac{1}{2}$ foot below equilibrium and released.

60. The weight is raised $\frac{2}{3}$ foot above equilibrium and released.

61. The weight is raised $\frac{2}{3}$ foot above equilibrium and released with an initial downward velocity of $\frac{1}{2}$ foot per second.

62. The weight is pulled $\frac{1}{2}$ foot below equilibrium and released with an initial upward velocity of $\frac{1}{2}$ foot per second.

63. The weight is pulled $\frac{1}{2}$ foot below equilibrium and released. The motion takes place in a medium that furnishes a damping force of magnitude $\frac{1}{8}|v|$ at all times.

64. The weight is pulled $\frac{1}{2}$ foot below equilibrium and released. The motion takes place in a medium that furnishes a damping force of magnitude $\frac{1}{4}|v|$ at all times.

65. Real Zeros The characteristic equation of the differential equation $y'' + ay' + by = 0$ has two equal real zeros given by $m = r$. Show that $y = C_1e^{rx} + C_2xe^{rx}$ is a solution.

66. Complex Zeros The characteristic equation of the differential equation

$$y'' + ay' + by = 0$$

has complex zeros given by $m_1 = \alpha + \beta i$ and $m_2 = \alpha - \beta i$. Show that $y = C_1e^{\alpha x}\cos\beta x + C_2e^{\alpha x}\sin\beta x$ is a solution.

True or False? **In Exercises 67–70, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

67. $y_1 = e^x$ and $y_2 = 3e^x$ are linearly dependent.

68. $y_1 = x$ and $y_2 = x^2$ are linearly dependent.

69. $y = x$ is a solution of

$$a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0$$

if and only if $a_1 = a_0 = 0$.

70. It is possible to choose a and b such that $y = x^2e^x$ is a solution of $y'' + ay' + by = 0$.

Wronskian **The *Wronskian* of two differentiable functions f and g, denoted by $W(f, g)$, is defined as the function given by the determinant**

$$W(f, g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}.$$

The functions f and g are linearly independent when there exists at least one value of x for which $W(f, g) \neq 0$. In Exercises 71–74, use the Wronskian to verify that the two functions are linearly independent.

71. $y_1 = e^{ax}$
$y_2 = e^{bx}, \quad a \neq b$

72. $y_1 = e^{ax}$
$y_2 = xe^{ax}$

73. $y_1 = e^{ax}\sin bx$
$y_2 = e^{ax}\cos bx, \quad b \neq 0$

74. $y_1 = x$
$y_2 = x^2$

Monty Rakusen/Cultura/Getty Images

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16.3 Second-Order Nonhomogeneous Linear Equations

- **Recognize the general solution of a second-order nonhomogeneous linear differential equation.**
- **Use the method of undetermined coefficients to solve a second-order nonhomogeneous linear differential equation.**
- **Use the method of variation of parameters to solve a second-order nonhomogeneous linear differential equation.**

SOPHIE GERMAIN (1776–1831)

Many of the early contributors to calculus were interested in forming mathematical models for vibrating strings and membranes, oscillating springs, and elasticity. One of these was the French mathematician Sophie Germain, who in 1816 was awarded a prize by the French Academy for a paper entitled "Memoir on the Vibrations of Elastic Plates."

See LarsonCalculus.com to read more of this biography.

Nonhomogeneous Equations

In the preceding section, damped oscillations of a spring were represented by the *homogeneous* second-order linear equation

$$\frac{d^2y}{dt^2} + \frac{p}{m}\left(\frac{dy}{dt}\right) + \frac{k}{m}y = 0. \qquad \text{Free motion}$$

This type of oscillation is called **free** because it is determined solely by the spring and gravity and is free of the action of other external forces. If such a system is also subject to an external periodic force, such as $a \sin bt$, caused by vibrations at the opposite end of the spring, then the motion is called **forced,** and it is characterized by the *nonhomogeneous* equation

$$\frac{d^2y}{dt^2} + \frac{p}{m}\left(\frac{dy}{dt}\right) + \frac{k}{m}y = a \sin bt. \qquad \text{Forced motion}$$

In this section, you will study two methods for finding the general solution of a nonhomogeneous linear differential equation. In both methods, the first step is to find the general solution of the corresponding homogeneous equation.

$$y = y_h \qquad \text{General solution of homogeneous equation}$$

Having done this, you try to find a particular solution of the nonhomogeneous equation.

$$y = y_p \qquad \text{Particular solution of nonhomogeneous equation}$$

By combining these two results, you can conclude that the general solution of the nonhomogeneous equation is

$$y = y_h + y_p \qquad \text{General solution of nonhomogeneous equation}$$

as stated in the next theorem.

THEOREM 16.5 Solution of Nonhomogeneous Linear Equation

Let

$$y'' + ay' + by = F(x)$$

be a second-order nonhomogeneous linear differential equation. If y_p is a particular solution of this equation and y_h is the general solution of the corresponding homogeneous equation, then

$$y = y_h + y_p$$

is the general solution of the nonhomogeneous equation.

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Method of Undetermined Coefficients

You already know how to find the solution y_h of a linear *homogeneous* differential equation. The remainder of this section looks at ways to find the particular solution y_p. When $F(x)$ in

$$y'' + ay' + by = F(x)$$

consists of sums or products of x^n, e^{mx}, $\cos \beta x$, or $\sin \beta x$, you can find a particular solution y_p by the method of **undetermined coefficients.** The object of this method is to guess that the solution y_p is a generalized form of $F(x)$. Here are some examples.

1. For $F(x) = 3x^2$, choose $y_p = Ax^2 + Bx + C$.
2. For $F(x) = 4xe^x$, choose $y_p = Axe^x + Be^x$.
3. For $F(x) = x + \sin 2x$, choose $y_p = (Ax + B) + C \sin 2x + D \cos 2x$.

Then, by substitution, determine the coefficients for the generalized solution.

EXAMPLE 1 Method of Undetermined Coefficients

Find the general solution of the equation $y'' - 2y' - 3y = 2 \sin x$.

Solution Note that the corresponding homogeneous equation is $y'' - 2y' - 3y = 0$. To find the general solution y_h, solve the characteristic equation as shown.

$$\begin{aligned} m^2 - 2m - 3 &= 0 \\ (m + 1)(m - 3) &= 0 \\ m &= -1, 3 \end{aligned}$$

So, the general solution of the homogeneous equation is $y_h = C_1e^{-x} + C_2e^{3x}$. Next, let y_p be a generalized form of $2 \sin x$.

$$\begin{aligned} y_p &= A \cos x + B \sin x \\ y_p' &= -A \sin x + B \cos x \\ y_p'' &= -A \cos x - B \sin x \end{aligned}$$

Substitution into the original differential equation yields

$$\begin{aligned} y'' - 2y' - 3y &= 2 \sin x \\ -A \cos x - B \sin x - 2(-A \sin x + B \cos x) - 3(A \cos x + B \sin x) &= 2 \sin x \\ -A \cos x - B \sin x + 2A \sin x - 2B \cos x - 3A \cos x - 3B \sin x &= 2 \sin x \\ (-4A - 2B) \cos x + (2A - 4B) \sin x &= 2 \sin x. \end{aligned}$$

By equating coefficients of like terms, you obtain the system of two equations

$$-4A - 2B = 0 \quad \text{and} \quad 2A - 4B = 2$$

with solutions

$$A = \frac{1}{5} \quad \text{and} \quad B = -\frac{2}{5}.$$

Therefore, a particular solution of the original nonhomogeneous equation is

$$y_p = \frac{1}{5} \cos x - \frac{2}{5} \sin x$$

Particular solution of nonhomogeneous equation

and the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= C_1e^{-x} + C_2e^{3x} + \frac{1}{5} \cos x - \frac{2}{5} \sin x. \end{aligned}$$

General solution of nonhomogeneous equation

In Example 1, the form of the homogeneous solution $y_h = C_1e^{-x} + C_2e^{3x}$ has no overlap with the function $F(x)$ in the equation $y'' + ay' + by = F(x)$. However, suppose the given differential equation in Example 1 were of the form

$$y'' - 2y' - 3y = e^{-x}.$$

Now it would make no sense to guess that the particular solution was

$$y = Ae^{-x}$$

because in the equation $y'' - 2y' - 3y = e^{-x}$, this solution yields

$$Ae^{-x} - 2(-Ae^{-x}) - 3Ae^{-x} = 0 \neq e^{-x}. \qquad y = Ae^{-x},\ y' = -Ae^{-x},\ y'' = Ae^{-x}$$

In such cases, you should alter your guess by multiplying by the lowest power of x that removes the duplication. For this particular problem, you would guess

$$y_p = Axe^{-x}.$$

Another case where you need to alter your guess for the particular solution y_p is shown in the next example.

EXAMPLE 2 Method of Undetermined Coefficients

Find the general solution of

$$y'' - 2y' = x + 2e^x.$$

Solution The corresponding homogeneous equation is $y'' - 2y' = 0$, and the characteristic equation

$$m^2 - 2m = 0$$
$$m(m - 2) = 0$$

has solutions $m = 0$ and $m = 2$. So,

$$y_h = C_1 + C_2e^{2x}. \qquad \text{General solution of homogeneous equation}$$

Because $F(x) = x + 2e^x$, your first choice for y_p would be $(A + Bx) + Ce^x$. However, because y_h *already* contains a constant term C_1, you should multiply the *polynomial part* $(A + Bx)$ by x and use

$$y_p = Ax + Bx^2 + Ce^x$$
$$y_p' = A + 2Bx + Ce^x$$
$$y_p'' = 2B + Ce^x.$$

Substitution into the original differential equation produces

$$y'' - 2y' = x + 2e^x$$
$$2B + Ce^x - 2(A + 2Bx + Ce^x) = x + 2e^x$$
$$(2B - 2A) - 4Bx - Ce^x = x + 2e^x.$$

Equating coefficients of like terms yields the system of three equations

$$2B - 2A = 0, \quad -4B = 1, \quad \text{and} \quad -C = 2$$

with solutions $A = B = -\frac{1}{4}$ and $C = -2$. Therefore,

$$y_p = -\frac{1}{4}x - \frac{1}{4}x^2 - 2e^x \qquad \text{Particular solution of nonhomogeneous equation}$$

and the general solution is

$$y = y_h + y_p$$
$$= C_1 + C_2e^{2x} - \frac{1}{4}x - \frac{1}{4}x^2 - 2e^x. \qquad \text{General solution of nonhomogeneous equation}$$

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In Example 2, the polynomial part of the initial guess $(A + Bx) + Ce^x$ for y_p overlapped by a constant term with

$$y_h = C_1 + C_2e^{2x}$$

and it was necessary to multiply the polynomial part by a power of x that removed the overlap. The next example further illustrates some choices for y_p that eliminate overlap with y_h. Remember that in all cases, the first guess for y_p should match the types of functions occurring in $F(x)$.

EXAMPLE 3 Choosing the Form of the Particular Solution

Determine a suitable choice for y_p for each differential equation, given its general solution of the homogeneous equation.

$y'' + ay' + by = F(x)$	y_h
a. $y'' = x^2$	$C_1 + C_2x$
b. $y'' + 2y' + 10y = 4 \sin 3x$	$C_1e^{-x} \cos 3x + C_2e^{-x} \sin 3x$
c. $y'' - 4y' + 4 = e^{2x}$	$C_1e^{2x} + C_2xe^{2x}$

Solution

a. Because $F(x) = x^2$, the normal choice for y_p would be $A + Bx + Cx^2$. However, because $y_h = C_1 + C_2x$ already contains a constant term and a linear term, you should multiply by x^2 to obtain

$$y_p = Ax^2 + Bx^3 + Cx^4.$$

b. Because $F(x) = 4 \sin 3x$ and each term in y_h contains a factor of e^{-x}, you can simply let

$$y_p = A \cos 3x + B \sin 3x.$$

c. Because $F(x) = e^{2x}$, the normal choice for y_p would be Ae^{2x}. However, because $y_h = C_1e^{2x} + C_2xe^{2x}$ already contains an e^{2x} term and an xe^{2x} term, you should multiply by x^2 to get

$$y_p = Ax^2e^{2x}.$$

EXAMPLE 4 Solving a Third-Order Equation

See LarsonCalculus.com for an interactive version of this type of example.

Find the general solution of $y''' + 3y'' + 3y' + y = x$.

Solution From Example 6 in Section 16.2, you know that the homogeneous solution is

$$y_h = C_1e^{-x} + C_2xe^{-x} + C_3x^2e^{-x}.$$

Because $F(x) = x$, let $y_p = A + Bx$ and obtain $y_p' = B$ and $y_p'' = 0$. So, by substitution into the original differential equation, you have

$$0 + 3(0) + 3(B) + A + Bx = x$$
$$(3B + A) + Bx = x.$$

Equating coefficients of like terms yields the system of two equations

$$3B + A = 0 \quad \text{and} \quad B = 1.$$

So, $B = 1$ and $A = -3$, which implies that $y_p = -3 + x$. Therefore, the general solution is

$$y = y_h + y_p$$
$$= C_1e^{-x} + C_2xe^{-x} + C_3x^2e^{-x} - 3 + x.$$

Variation of Parameters

The method of undetermined coefficients works well when $F(x)$ is made up of polynomials or functions whose successive derivatives have a cyclical pattern. For functions such as $1/x$ and $\tan x$, which do not have such characteristics, it is better to use a more general method called **variation of parameters.** In this method, you assume that y_p has the same *form* as y_h, except that the constants in y_h are replaced by variables.

Variation of Parameters

To find the general solution of the equation $y'' + ay' + by = F(x)$, use these steps.

1. Find $y_h = C_1y_1 + C_2y_2$.
2. Replace the constants by variables to form $y_p = u_1y_1 + u_2y_2$.
3. Solve the following system for u_1' and u_2'.

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= F(x) \end{aligned}$$

4. Integrate to find u_1 and u_2. The general solution is $y = y_h + y_p$.

EXAMPLE 5 **Variation of Parameters**

Solve the differential equation

$$y'' - 2y' + y = \frac{e^x}{2x}, \quad x > 0.$$

Solution The characteristic equation

$$m^2 - 2m + 1 = 0 \quad \Rightarrow \quad (m - 1)^2 = 0$$

has one repeated solution, $m = 1$. So, the homogeneous solution is

$$y_h = C_1y_1 + C_2y_2 = C_1e^x + C_2xe^x.$$

Replacing C_1 and C_2 by u_1 and u_2 produces

$$y_p = u_1y_1 + u_2y_2 = u_1e^x + u_2xe^x.$$

The resulting system of equations is

$$\begin{aligned} u_1'e^x + u_2'xe^x &= 0 \\ u_1'e^x + u_2'(xe^x + e^x) &= \frac{e^x}{2x}. \end{aligned}$$

Subtracting the second equation from the first produces $u_2' = 1/(2x)$. Then, by substitution in the first equation, you have $u_1' = -\frac{1}{2}$. Finally, integration yields

$$u_1 = -\int \frac{1}{2}\,dx = -\frac{x}{2} \quad \text{and} \quad u_2 = \frac{1}{2}\int \frac{1}{x}\,dx = \frac{1}{2}\ln x = \ln\sqrt{x}.$$

From this result, it follows that a particular solution is

$$y_p = -\frac{1}{2}xe^x + \left(\ln\sqrt{x}\right)xe^x$$

and the general solution is

$$y = C_1e^x + C_2xe^x - \frac{1}{2}xe^x + xe^x\ln\sqrt{x}.$$

Exploration

Notice in Example 5 that the constants of integration were not introduced when finding u_1 and u_2. Show that for

$$u_1 = -\frac{x}{2} + a_1 \quad \text{and} \quad u_2 = \ln \sqrt{x} + a_2$$

the general solution

$$y = y_h + y_p = C_1 e^x + C_2 e^x - \frac{1}{2}xe^x + xe^x \ln \sqrt{x}$$

yields the same result as the solution obtained in the example.

EXAMPLE 6 **Variation of Parameters**

Solve the differential equation $y'' + y = \tan x$.

Solution Because the characteristic equation $m^2 + 1 = 0$ has solutions $m = \pm i$, the homogeneous solution is $y_h = C_1 \cos x + C_2 \sin x$. Replacing C_1 and C_2 by u_1 and u_2 produces $y_p = u_1 \cos x + u_2 \sin x$. The resulting system of equations is

$$\begin{aligned} u_1' \cos x + u_2' \sin x &= 0 \\ -u_1' \sin x + u_2' \cos x &= \tan x. \end{aligned}$$

Multiplying the first equation by $\sin x$ and the second by $\cos x$ produces

$$\begin{aligned} u_1' \sin x \cos x + u_2' \sin^2 x &= 0 \\ -u_1' \sin x \cos x + u_2' \cos^2 x &= \sin x. \end{aligned}$$

Adding these two equations produces $u_2' = \sin x$, which implies that

$$\begin{aligned} u_1' \sin x \cos x + (\sin x) \sin^2 x &= 0 \\ u_1' \sin x \cos x &= -\sin^3 x \\ u_1' &= -\frac{\sin^2 x}{\cos x} \\ u_1' &= \frac{\cos^2 x - 1}{\cos x} \\ u_1' &= \cos x - \sec x. \end{aligned}$$

Integration yields

$$u_1 = \int (\cos x - \sec x)\, dx = \sin x - \ln|\sec x + \tan x|$$

and

$$u_2 = \int \sin x\, dx = -\cos x$$

so that the particular solution is

$$\begin{aligned} y_p &= \sin x \cos x - \cos x \ln|\sec x + \tan x| - \sin x \cos x \\ &= -\cos x \ln|\sec x + \tan x| \end{aligned}$$

and the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= C_1 \cos x + C_2 \sin x - \cos x \ln|\sec x + \tan x|. \end{aligned}$$

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16.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Writing** What is the form of the general solution of a second-order nonhomogeneous linear differential equation?

2. **Choosing a Method** Determine whether you would use the method of undetermined coefficients or the method of variation of parameters to find the general solution of each differential equation. Explain your reasoning. (Do not solve the equations.)

(a) $y'' + 3y' + y = x^2$ (b) $y'' + y = \csc x$

(c) $y''' - 6y = 3x - e^{-2x}$

Method of Undetermined Coefficients In Exercises 3–6, solve the differential equation by the method of undetermined coefficients.

3. $y'' + 7y' + 12y = 3x + 1$
4. $y'' - y' - 6y = 4$
5. $y'' - 8y' + 16y = e^{3x}$
6. $y'' - 2y' - 15y = \sin x$

Choosing the Form of the Particular Solution In Exercises 7–10, determine a suitable choice for y_p for the differential equation, given its general solution of the homogeneous equation. Explain your reasoning. (Do not solve the equation.)

$y'' + ay' + by = F(x)$	y_h
7. $y'' + y' = 4x + 6$	$C_1 + C_2e^{-x}$
8. $y'' - 9y = x + 2e^{-3x}$	$C_1e^{-3x} + C_2e^{3x}$
9. $3y'' + 6y' = 4 + \sin x$	$C_1 + C_2e^{-2x}$
10. $y'' + y = 8\cos x$	$C_1\cos x + C_2\sin x$

Method of Undetermined Coefficients In Exercises 11–16, solve the differential equation by the method of undetermined coefficients.

11. $y'' + 2y' = e^{-2x}$
12. $y'' - 9y = 5e^{3x}$
13. $y'' + 9y = \sin 3x$
14. $16y'' - 8y' + y = 4(x + e^{x/4})$
15. $y''' - 3y' + 4y = 2 + e^{2x}$
16. $y''' - 3y' + 2y = 2e^{-2x}$

Using Initial Conditions In Exercises 17–22, solve the differential equation by the method of undetermined coefficients subject to the initial condition(s).

17. $y'' + y = x^3$
$y(0) = 1,\ y'(0) = 0$
18. $y'' + 4y = 4$
$y(0) = 1,\ y'(0) = 6$
19. $y'' + y' = 2\sin x$
$y(0) = 0,\ y'(0) = -3$
20. $y'' + y' - 2y = 3\cos 2x$
$y(0) = -1,\ y'(0) = 2$
21. $y' - 4y = xe^x - xe^{4x}$
$y(0) = \frac{1}{3}$
22. $y' + 2y = \sin x$
$y\left(\frac{\pi}{2}\right) = \frac{2}{5}$

Method of Variation of Parameters In Exercises 23–28, solve the differential equation by the method of variation of parameters.

23. $y'' + y = \sec x$
24. $y'' + y = \sec x \tan x$
25. $y'' + 4y = \csc 2x$
26. $y'' - 4y' + 4y = x^2e^{2x}$
27. $y'' - 2y' + y = e^x \ln x$
28. $y'' - 4y' + 4y = \dfrac{e^{2x}}{x}$

Electrical Circuits In Exercises 29 and 30, use the electrical circuit differential equation

$$\frac{d^2q}{dt^2} + \left(\frac{R}{L}\right)\frac{dq}{dt} + \left(\frac{1}{LC}\right)q = \left(\frac{1}{L}\right)E(t)$$

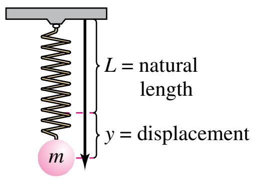
where R is the resistance (in ohms), C is the capacitance (in farads), L is the inductance (in henrys), $E(t)$ is the electromotive force (in volts), and q is the charge on the capacitor (in coulombs). Find the charge q as a function of time t for the electrical circuit described. Assume that $q(0) = 0$ and $q'(0) = 0$.

29. $R = 20,\ C = 0.02,\ L = 2,\ E(t) = 12\sin 5t$
30. $R = 20,\ C = 0.02,\ L = 1,\ E(t) = 10\sin 5t$

Motion of a Spring In Exercises 31–34, use the differential equation

$$\frac{w}{g}y''(t) + by'(t) + ky(t) = \frac{w}{g}F(t)$$

which models the oscillating motion of an object on the end of a spring (see figure). In the equation, y is the displacement from equilibrium (positive direction is downward), measured in feet, t is time in seconds, w is the weight of the object, g is the acceleration due to gravity, b is the magnitude of the resistance to the motion, k is the spring constant from Hooke's Law, and $F(t)$ is the acceleration imposed on the system. Find the displacement y as a function of time t for the oscillating motion described subject to the initial conditions. Use a graphing utility to graph the displacement function.



Spring displacement

31. $w = 24,\ g = 32,\ b = 0,\ k = 48,\ F(t) = 48\sin 4t$
$y(0) = \frac{1}{4},\ y'(0) = 0$

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32. $w = 2,\ g = 32,\ b = 0,\ k = 4,\ F(t) = 4 \sin 8t$

$y(0) = \frac{1}{4},\ y'(0) = 0$

33. $w = 2,\ g = 32,\ b = 1,\ k = 4,\ F(t) = 4 \sin 8t$

$y(0) = \frac{1}{4},\ y'(0) = -3$

34. $w = 4,\ g = 32,\ b = \frac{1}{2},\ k = \frac{25}{2},\ F(t) = 0$

$y(0) = \frac{1}{2},\ y'(0) = -4$

EXPLORING CONCEPTS

35. Motion of a Spring Rewrite y_h in the solution to Exercise 31 by using the identity

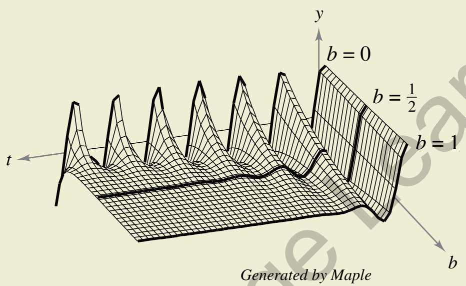
$$a \cos \omega t + b \sin \omega t = \sqrt{a^2 + b^2} \sin(\omega t + \phi)$$

where $\phi = \arctan a/b$.

36. HOW DO YOU SEE IT? The figure shows the particular solution of the differential equation

$$\frac{4}{32}y'' + by' + \frac{25}{2}y = 0$$

that models the oscillating motion of an object on the end of a spring and satisfies the initial conditions $y(0) = \frac{1}{2}$ and $y'(0) = -4$ for values of the resistance component b in the interval $[0, 1]$. According to the figure, is the motion damped or undamped when $b = 0$? When $b > 0$? (You do not need to solve the differential equation.)



37. Motion of a Spring Refer to the differential equation and the initial conditions given in Exercise 36.

(a) When there is no resistance to the motion ($b = 0$), describe the motion.

(b) For $b > 0$, what is the ultimate effect of the retarding force?

(c) Is there a real number M such that there will be no oscillations of the spring for $b > M$? Explain your answer.

38. Solving a Differential Equation Solve the differential equation given that y_1 and y_2 are solutions of the corresponding homogeneous equation.

(a) $x^2y'' - xy' + y = 4x \ln x$

$y_1 = x,\ y_2 = x \ln x$

(b) $x^2y'' + xy' + 4y = \sin(\ln x)$

$y_1 = \sin(\ln x^2),\ y_2 = \cos(\ln x^2)$

True or False? **In Exercises 39 and 40, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

39. $y_p = -e^{2x} \cos e^{-x}$ is a particular solution of the differential equation

$$y'' - 3y' + 2y = \cos e^{-x}$$

40. $y_p = -\frac{1}{8}e^{2x}$ is a particular solution of the differential equation

$$y'' - 6y' = e^{2x}.$$

PUTNAM EXAM CHALLENGE

41. For all real x, the real-valued function $y = f(x)$ satisfies $y'' - 2y' + y = 2e^x$.

(a) If $f(x) > 0$ for all real x, must $f'(x) > 0$ for all real x? Explain.

(b) If $f'(x) > 0$ for all real x, must $f(x) > 0$ for all real x? Explain.

This problem was composed by the Committee on the Putnam Prize Competition.

SECTION PROJECT

Parachute Jump

The fall of a parachutist is described by the second-order linear differential equation

$$\left(-\frac{w}{g}\right)\frac{d^2y}{dt^2} - k\frac{dy}{dt} = w$$

where w is the weight of the parachutist, y is the height at time t, g is the acceleration due to gravity, and k is the drag factor of the parachute.

(a) The parachute is opened at 2000 feet, so

$$y(0) = 2000.$$

At that time, the velocity is

$$y'(0) = -100 \text{ feet per second.}$$

For a 160-pound parachutist who has a parachute with a drag factor of $k = 8$, the differential equation is

$$-5y'' - 8y' = 160.$$

Using the initial conditions, verify that the solution of the differential equation is

$$y = 1950 + 50e^{-1.6t} - 20t.$$

(b) Consider a 192-pound parachutist who has a parachute with a drag factor of $k = 9$. Using the initial conditions given in part (a), write and solve a differential equation that describes the fall of the parachutist.

16.4 Series Solutions of Differential Equations

- Use a power series to solve a differential equation.
- Use a Taylor series to find the series solution of a differential equation.

Power Series Solution of a Differential Equation

Power series can be used to solve certain types of differential equations. This section begins with the general **power series solution** method.

Recall from Chapter 9 that a power series represents a function f on an interval of convergence and that you can successively differentiate the power series to obtain a series for f', f'', and so on. These properties are used in the power series solution method demonstrated in the first two examples.

EXAMPLE 1 **Power Series Solution**

Use a power series to solve the differential equation $y' - 2y = 0$.

Solution Assume that $y = \sum_{n=0}^{\infty} a_n x^n$ is a solution. Then,

$$y' = \sum_{n=1}^{\infty} na_n x^{n-1}.$$

Substituting for y' and $-2y$, you obtain the following series form of the differential equation. (Note that, from the third step to the fourth, the index of summation is changed to ensure that x^n occurs in both sums.)

$$y' - 2y = 0$$

$$\sum_{n=1}^{\infty} na_n x^{n-1} - 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} 2a_n x^n$$

$$\sum_{n=0}^{\infty} (n+1)a_{n+1} x^n = \sum_{n=0}^{\infty} 2a_n x^n$$

Now, by equating coefficients of like terms, you obtain the **recursion formula**

$$(n+1)a_{n+1} = 2a_n$$

which implies that

$$a_{n+1} = \frac{2a_n}{n+1}, \quad n \geq 0.$$

This formula generates the following results.

a_0	a_1	a_2	a_3	a_4	a_5	. . .
a_0	$2a_0$	$\frac{2^2 a_0}{2}$	$\frac{2^3 a_0}{3!}$	$\frac{2^4 a_0}{4!}$	$\frac{2^5 a_0}{5!}$	. . .

Using these values as the coefficients for the *solution* series, you have

$$y = \sum_{n=0}^{\infty} \frac{2^n a_0}{n!} x^n$$

$$= a_0 \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$$

$$= a_0 e^{2x}.$$

Exploration

In Example 1, the differential equation could be solved easily without using a series. Determine which method should be used to solve the differential equation

$$y' - 2y = 0$$

and show that the result is the same as that obtained in the example.

In Example 1, the differential equation could be solved easily without using a series. The differential equation in Example 2 cannot be solved by any of the methods discussed in previous sections.

EXAMPLE 2 Power Series Solution

Use a power series to solve the differential equation

$$y'' + xy' + y = 0.$$

Solution Assume that $y = \sum_{n=0}^{\infty} a_n x^n$ is a solution. Then you have

$$y' = \sum_{n=1}^{\infty} na_n x^{n-1}, \quad xy' = \sum_{n=1}^{\infty} na_n x^n, \quad y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

Substituting for y'', xy', and y in the given differential equation, you obtain the following series.

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} na_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = -\sum_{n=0}^{\infty} (n+1)a_n x^n$$

To obtain equal powers of x, adjust the summation indices by replacing n by $n + 2$ in the left-hand sum, to obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n = -\sum_{n=0}^{\infty} (n+1)a_n x^n.$$

By equating coefficients, you have

$$(n+2)(n+1)a_{n+2} = -(n+1)a_n$$

from which you obtain the recursion formula

$$a_{n+2} = -\frac{(n+1)}{(n+2)(n+1)}a_n = -\frac{a_n}{n+2}, \quad n \geq 0,$$

and the coefficients of the solution series are as follows.

$$a_2 = -\frac{a_0}{2} \qquad a_3 = -\frac{a_1}{3}$$

$$a_4 = -\frac{a_2}{4} = \frac{a_0}{2 \cdot 4} \qquad a_5 = -\frac{a_3}{5} = \frac{a_1}{3 \cdot 5}$$

$$a_6 = -\frac{a_4}{6} = -\frac{a_0}{2 \cdot 4 \cdot 6} \qquad a_7 = -\frac{a_5}{7} = -\frac{a_1}{3 \cdot 5 \cdot 7}$$

$$\vdots \qquad \vdots$$

$$a_{2k} = \frac{(-1)^k a_0}{2 \cdot 4 \cdot 6 \cdots (2k)} = \frac{(-1)^k a_0}{2^k(k!)} \qquad a_{2k+1} = \frac{(-1)^k a_1}{3 \cdot 5 \cdot 7 \cdots (2k+1)}$$

So, you can represent the general solution as the sum of two series—one for the even-powered terms with coefficients in terms of a_0, and one for the odd-powered terms with coefficients in terms of a_1.

$$y = a_0\left(1 - \frac{x^2}{2} + \frac{x^4}{2 \cdot 4} - \cdots\right) + a_1\left(x - \frac{x^3}{3} + \frac{x^5}{3 \cdot 5} - \cdots\right)$$

$$= a_0 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^k(k!)} + a_1 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{3 \cdot 5 \cdot 7 \cdots (2k+1)}$$

The solution has two arbitrary constants, a_0 and a_1, as you would expect in the general solution of a second-order differential equation.

Approximation by Taylor Series

A second type of series solution method involves a differential equation with *initial conditions* and makes use of Taylor series, as given in Section 9.10.

EXAMPLE 3 Approximation by Taylor Series

See LarsonCalculus.com for an interactive version of this type of example.

Use a Taylor series to find the first six terms of the series solution of

$$y' = y^2 - x$$

for the initial condition $y = 1$ when $x = 0$. Then use this polynomial to approximate values of y for $0 \le x \le 1$.

Solution Recall from Section 9.10 that, for $c = 0$,

$$y = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \cdots.$$

Because $y(0) = 1$ and $y' = y^2 - x$, you obtain the following.

$$\begin{aligned} & & y(0) &= 1 \\ y' &= y^2 - x & y'(0) &= 1 \\ y'' &= 2yy' - 1 & y''(0) &= 2 - 1 = 1 \\ y''' &= 2yy'' + 2(y')^2 & y'''(0) &= 2 + 2 = 4 \\ y^{(4)} &= 2yy''' + 6y'y'' & y^{(4)}(0) &= 8 + 6 = 14 \\ y^{(5)} &= 2yy^{(4)} + 8y'y''' + 6(y'')^2 & y^{(5)}(0) &= 28 + 32 + 6 = 66 \end{aligned}$$

So, y can be approximated by the first six terms of the series solution shown below.

$$\begin{aligned} y &\approx y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \frac{y^{(5)}(0)}{5!}x^5 \\ &= 1 + x + \frac{1}{2}x^2 + \frac{4}{3!}x^3 + \frac{14}{4!}x^4 + \frac{66}{5!}x^5 \end{aligned}$$

Using this polynomial, you can approximate values for y in the interval $0 \le x \le 1$, as shown in the table below.

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y	1.0000	1.1057	1.2264	1.3691	1.5432	1.7620	2.0424	2.4062	2.8805	3.4985	4.3000

In addition to approximating values of a function, you can also use a series solution to sketch a graph. In Figure 16.7, the series solutions of $y' = y^2 - x$ using the first two, four, and six terms are shown, along with an approximation found using a computer algebra system. The approximations are nearly the same for values of x close to 0. As x approaches 1, however, there is a noticeable difference among the approximations. For a series solution that is more accurate near $x = 1$, repeat Example 3 using $c = 1$.

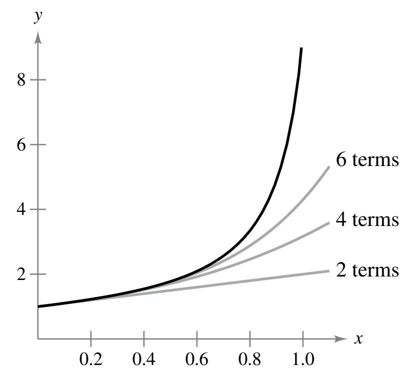


Figure 16.7

16.4 Exercises

CONCEPT CHECK

1. **Power Series Solution Method** Describe how to use power series to solve a differential equation.
2. **Recursion Formula** What is a recursion formula? Give an example.

Power Series Solution **In Exercises 3–6, use a power series to solve the differential equation.**

3. $5y' + y = 0$
4. $(x + 2)y' + y = 0$
5. $y' + 3xy = 0$
6. $y' - 2xy = 0$

Finding Terms of a Power Series Solution **In Exercises 7 and 8, the solution of the differential equation is a sum of two power series. Find the first three terms of each power series. (See Example 2.)**

7. $(x^2 + 4)y'' + y = 0$
8. $y'' + x^2y = 0$

Approximation by Taylor Series **In Exercises 9–14, use a Taylor series to find the first n terms of the series solution of the differential equation that satisfies the initial condition(s). Use this polynomial to approximate y for the given value of x.**

9. $y' + (2x - 1)y = 0,\ y(0) = 2,\ n = 5,\ x = \frac{1}{2}$
10. $y' - 2xy = 0,\ y(0) = 1,\ n = 4,\ x = 1$
11. $y'' - 2xy = 0,\ y(0) = 1,\ y'(0) = -3,\ n = 6,\ x = \frac{1}{4}$
12. $y'' - 2xy' + y = 0,\ y(0) = 1,\ y'(0) = 2,\ n = 8,\ x = \frac{1}{2}$
13. $y'' + x^2y' - (\cos x)y = 0,\ y(0) = 3,\ y'(0) = 2,\ n = 4,\ x = \frac{1}{3}$
14. $y'' + e^xy' - (\sin x)y = 0,\ y(0) = -2,\ y'(0) = 1,\ n = 4,\ x = \frac{1}{5}$

EXPLORING CONCEPTS

Using Different Methods **In Exercises 15–18, verify that the power series solution of the differential equation is equivalent to the solution found using previously learned solution techniques.**

15. $y' - ky = 0$
16. $y' + ky = 0$
17. $y'' - k^2y = 0$
18. $y'' + k^2y = 0$

19. **Investigation** Consider the differential equation

$$y'' - xy' = 0$$

with the initial conditions

$$y(0) = 0 \quad \text{and} \quad y'(0) = 2.$$

(a) Find the series solution satisfying the initial conditions.

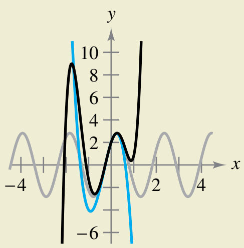
(b) Use a graphing utility to graph the third-degree and fifth-degree series approximations of the solution. Identify the approximations.

(c) Identify the symmetry of the solution.

20. **HOW DO YOU SEE IT?** Consider the differential equation

$$y'' + 9y = 0$$

with initial conditions $y(0) = 2$ and $y'(0) = 6$. The figure shows the graph of the solution of the differential equation and the third-degree and fifth-degree polynomial approximations of the solution. Identify each.



Verifying that a Series Converges **In Exercises 21–24, use the power series solution of the differential equation to verify that the series converges to the given function on the indicated interval.**

21. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x,\ (-\infty, \infty)$

Differential equation: $y' - y = 0$

22. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \cos x,\ (-\infty, \infty)$

Differential equation: $y'' + y = 0$

23. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n + 1} = \arctan x,\ (-1, 1)$

Differential equation: $(x^2 + 1)y'' + 2xy' = 0$

24. $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!x^{2n+1}}{(2^n n!)^2(2n + 1)} = \arcsin x,\ (-1, 1)$

Differential equation: $(1 - x^2)y'' - xy' = 0$

25. **Airy's Equation** Find the first six terms of the series solution of Airy's equation, $y'' - xy = 0$.

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Review Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Testing for Exactness In Exercises 1 and 2, determine whether the differential equation is exact.

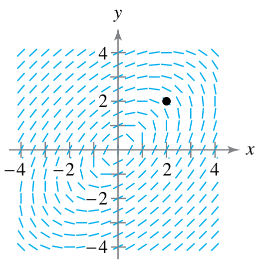
1. $(y + x^3 + xy^2)\,dx - x\,dy = 0$
2. $(5x - y)\,dx + (5y - x)\,dy = 0$

Solving an Exact Differential Equation In Exercises 3–6, verify that the differential equation is exact. Then find the general solution.

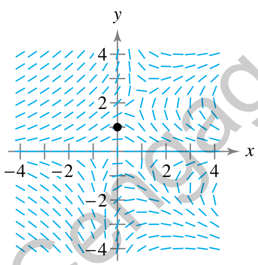
3. $(10x + 8y + 2)\,dx + (8x + 5y + 2)\,dy = 0$
4. $(2x - 2y^3 + y)\,dx + (x - 6xy^2)\,dy = 0$
5. $(x - y - 5)\,dx - (x + 3y - 2)\,dy = 0$
6. $y \sin(xy)\,dx + [x \sin(xy) + y]\,dy = 0$

Graphical and Analytic Analysis In Exercises 7 and 8, (a) sketch an approximate solution of the differential equation satisfying the initial condition on the slope field, (b) find the particular solution that satisfies the initial condition, and (c) use a graphing utility to graph the particular solution. Compare the graph with the sketch in part (a).

7. $(2x - y)\,dx + (2y - x)\,dy = 0,\ \ y(2) = 2$



8. $(6xy - y^3)\,dx + (4y + 3x^2 - 3xy^2)\,dy = 0,\ \ y(0) = 1$



Finding a Particular Solution In Exercises 9–12, find the particular solution of the differential equation that satisfies the initial condition.

9. $(2x + y - 3)\,dx + (x - 3y + 1)\,dy = 0,\ \ y(2) = 0$
10. $3x^2y^2\,dx + (2x^3y - 3y^2)\,dy = 0,\ \ y(1) = 2$
11. $-\cos 2y\,dx + 2x \sin 2y\,dy = 0,\ \ y(3) = \pi$
12. $[9 + \ln(xy^3)]\,dx + \dfrac{3x}{y}\,dy = 0,\ \ y(1) = 1$

Finding an Integrating Factor In Exercises 13–16, find the integrating factor that is a function of x or y alone and use it to find the general solution of the differential equation.

13. $(3x^2 - y^2)\,dx + 2xy\,dy = 0$
14. $2xy\,dx + (y^2 - x^2)\,dy = 0$
15. $dx + (3x - e^{-2y})\,dy = 0$
16. $\cos y\,dx - [2(x - y) \sin y + \cos y]\,dy = 0$

Verifying a Solution In Exercises 17 and 18, verify the solution of the differential equation. Then use a graphing utility to graph the particular solutions for several different values of C_1 and C_2.

	Solution	Differential Equation
17.	$y = C_1e^{-3x} + C_2e^{2x}$	$y'' + y' - 6y = 0$
18.	$y = C_1 \cos 3x + C_2 \sin 3x$	$y'' + 9y = 0$

Finding a General Solution In Exercises 19–28, use a characteristic equation to find the general solution of the linear differential equation.

19. $2y'' + 5y' + 3y = 0$
20. $y'' - 4y' - 2y = 0$
21. $y'' - 6y' = 0$
22. $25y'' + 30y' + 9y = 0$
23. $y'' + 8y = 0$
24. $y'' + y' + 3y = 0$
25. $y''' - 2y'' - 3y' = 0$
26. $y''' - 6y'' + 12y' - 8y = 0$
27. $y^{(4)} - 5y'' = 0$
28. $y^{(4)} + 6y'' + 9y = 0$

Finding a Particular Solution: Initial Conditions In Exercises 29–32, use a characteristic equation to find the particular solution of the linear differential equation that satisfies the initial conditions.

29. $y'' - y' - 2y = 0$
    $y(0) = 0,\ y'(0) = 3$
30. $y'' + 4y' + 5y = 0$
    $y(0) = 2,\ y'(0) = -7$
31. $y'' + 2y' - 3y = 0$
    $y(0) = 2,\ y'(0) = 0$
32. $y'' + 12y' + 36y = 0$
    $y(0) = 2,\ y'(0) = 1$

Finding a Particular Solution: Boundary Conditions In Exercises 33 and 34, use a charcteristic equation to find the particular solution of the linear differential equation that satisfies the boundary conditions.

33. $y'' + 2y' + 5y = 0$
    $y(1) = 4,\ y(2) = 0$
34. $y'' + y = 0$
    $y(0) = 2,\ y(\pi/2) = 1$

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Motion of a Spring **In Exercises 35 and 36, a 64-pound weight stretches a spring $\frac{4}{3}$ feet from its natural length. Use the given information to find a formula for the position of the weight as a function of time.**

35. The weight is pulled $\frac{1}{2}$ foot below equilibrium and released.

36. The weight is pulled $\frac{3}{4}$ foot below equilibrium and released. The motion takes place in a medium that furnishes a damping force of magnitude $\frac{1}{8}|v|$ at all times.

Method of Undetermined Coefficients **In Exercises 37–40, solve the differential equation by the method of undetermined coefficients.**

37. $y'' + y = x^3 + x$

38. $y'' + 2y = e^{2x} + x$

39. $y'' - 8y' - 9y = 9x - 10$

40. $y'' + 5y' + 4y = x^2 + \sin 2x$

Choosing the Form of the Particular Solution **In Exercises 41 and 42, determine a suitable choice for y_p for the differential equation, given its general solution of the homogeneous equation. Explain your reasoning. (Do not solve the equation.)**

	$y'' + ay' + by = F(x)$	y_h
41.	$y'' - 4y' + 3y = e^x + 8e^{3x}$	$C_1e^x + C_2e^{3x}$
42.	$y'' = 2x + 1$	$C_1 + C_2x$

Method of Undetermined Coefficients **In Exercises 43 and 44, solve the differential equation by the method of undetermined coefficients.**

43. $y'' + y = 2\cos x$

44. $2y'' - y' = 4x$

Using Initial Conditions **In Exercises 45–50, solve the differential equation by the method of undetermined coefficients subject to the initial conditions.**

45. $y'' - y' - 6y = 54$
$y(0) = 2,\ y'(0) = 0$

46. $y'' + 25y = e^x$
$y(0) = 0,\ y'(0) = 0$

47. $y'' + 4y = \cos x$
$y(0) = 6,\ y'(0) = -6$

48. $y'' + 3y' = 6x$
$y(0) = 2,\ y'(0) = \frac{10}{3}$

49. $y'' - y' - 2y = 1 + xe^{-x}$
$y(0) = 1,\ y'(0) = 3$

50. $y''' - y'' = 4x^2$
$y(0) = 1,\ y'(0) = 1,\ y''(0) = 1$

Method of Variation of Parameters **In Exercises 51–54, solve the differential equation by the method of variation of parameters.**

51. $y'' + 9y = \csc 3x$

52. $4y'' + y = \sec\frac{x}{2}\tan\frac{x}{2}$

53. $y'' - 2y' + y = 2xe^x$

54. $y'' + 2y' + y = \frac{1}{x^2e^x}$

55. **Electrical Circuit** The differential equation

$$\frac{d^2q}{dt^2} + 4\frac{dq}{dt} + 8q = 3\sin 4t$$

models the charge q on a capacitor of an electrical circuit. Find the charge q as a function of time t. Assume that $q(0) = 0$ and $q'(0) = 0$.

56. **Investigation** The differential equation

$$\frac{8}{32}y'' + by' + ky = \frac{8}{32}F(t), \quad y(0) = \frac{1}{2}, \quad y'(0) = 0$$

models the oscillating motion of an object on the end of a spring, where y is the displacement from equilibrium (positive direction is downward), measured in feet, t is time in seconds, b is the magnitude of the resistance to the motion, k is the spring constant from Hooke's Law, and $F(t)$ is the acceleration imposed on the system.

(a) Solve the differential equation and use a graphing utility to graph the solution for each of the assigned quantities for b, k, and $F(t)$.

(i) $b = 0,\ k = 1,\ F(t) = 24\sin\pi t$

(ii) $b = 0,\ k = 2,\ F(t) = 24\sin(2\sqrt{2}t)$

(iii) $b = 0.1,\ k = 2,\ F(t) = 0$

(iv) $b = 1,\ k = 2,\ F(t) = 0$

(b) Describe the effect of increasing the resistance to motion b.

(c) Explain how the motion of the object changes when a stiffer spring (greater value of k) is used.

57. **Think About It**

(a) Explain how, by observation, you know that a form of a particular solution of the differential equation $y'' + 3y = 12\sin x$ is $y_p = A\sin x$.

(b) Use your explanation in part (a) to find a particular solution of the differential equation $y'' + 5y = 10\cos x$.

(c) Compare the algebra required to find particular solutions in parts (a) and (b) with that required when the form of the particular solution is $y_p = A\cos x + B\sin x$.

58. **Think About It** Explain how you can find a particular solution of the differential equation $y'' + 4y' + 6y = 30$ by observation.

Power Series Solution **In Exercises 59 and 60, use a power series to solve the differential equation.**

59. $(x - 4)y' + y = 0$

60. $y'' + 3xy' - 3y = 0$

Approximation by Taylor Series **In Exercises 61 and 62, use a Taylor series to find the first n terms of the series solution of the differential equation that satisfies the initial conditions. Use this polynomial to approximate y for the given value of x.**

61. $y'' + y' - e^xy = 0,\ y(0) = 2,\ y'(0) = 0,\ n = 4,\ x = \frac{1}{4}$

62. $y'' + xy = 0,\ y(0) = 1,\ y'(0) = 1,\ n = 6,\ x = \frac{1}{2}$

P.S. Problem Solving

1. Finding a General Solution Find the value of k that makes the differential equation

$$(3x^2 + kxy^2)\,dx - (5x^2y + ky^2)\,dy = 0$$

exact. Using this value of k, find the general solution.

2. Using an Integrating Factor The differential equation $(kx^2 + y^2)\,dx - kxy\,dy = 0$ is not exact, but the integrating factor $1/x^2$ makes it exact.

(a) Use this information to find the value of k.

(b) Using this value of k, find the general solution.

3. Finding a General Solution Find the general solution of the differential equation $y'' - a^2y = 0$, $a > 0$. Show that the general solution can be written in the form

$$y = C_1 \cosh ax + C_2 \sinh ax.$$

4. Finding a General Solution Find the general solution of the differential equation $y'' + \beta^2 y = 0$. Show that the general solution can be written in the form

$$y = C \sin(\beta x + \phi),\quad 0 \le \phi < 2\pi.$$

5. Distinct Real Zeros Given that the characteristic equation of the differential equation $y'' + ay' + by = 0$ has two distinct real zeros, $m_1 = r + s$ and $m_2 = r - s$, where r and s are real numbers, show that the general solution of the differential equation can be written in the form

$$y = e^{rx}(C_1 \cosh sx + C_2 \sinh sx).$$

6. Limit of a Solution Given that a and b are positive and that $y(x)$ is a solution of the differential equation

$$y'' + ay' + by = 0$$

show that $\lim_{x\to\infty} y(x) = 0$.

7. Trivial and Nontrivial Solutions Consider the differential equation $y'' + ay = 0$ with boundary conditions $y(0) = 0$ and $y(L) = 0$ for some nonzero real number L.

(a) For $a = 0$, show that the differential equation has only the trivial solution $y = 0$.

(b) For $a < 0$, show that the differential equation has only the trivial solution $y = 0$.

(c) For $a > 0$, find the value(s) of a for which the solution is nontrivial. Then find the corresponding solution(s).

8. Euler's Differential Equation **Euler's differential equation** is of the form

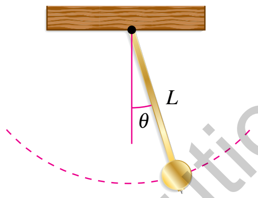
$$x^2y'' + axy' + by = 0,\quad x > 0$$

where a and b are constants.

(a) Show that this equation can be transformed into a second-order linear differential equation with constant coefficients by using the substitution $x = e^t$.

(b) Solve $x^2y'' + 6xy' + 6y = 0$.

9. Pendulum Consider a pendulum of length L that swings by the force of gravity only.



For small values of $\theta = \theta(t)$, the motion of the pendulum can be approximated by the differential equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

where g is the acceleration due to gravity.

(a) Find the general solution of the differential equation and show that it can be written in the form

$$\theta(t) = A\cos\left[\sqrt{\frac{g}{L}}(t + \phi)\right].$$

(b) Find the particular solution for a pendulum of length 0.25 meter when the initial conditions are $\theta(0) = 0.1$ radian and $\theta'(0) = 0.5$ radian per second. (Use $g = 9.8$ meters per second per second.)

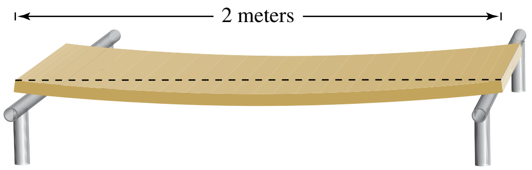
(c) Determine the period of the pendulum.

(d) Determine the maximum value of θ.

(e) How much time from $t = 0$ does it take for θ to be 0 the first time? the second time?

(f) What is the angular velocity θ' when $\theta = 0$ the first time? the second time?

10. Deflection of a Beam A horizontal beam with a length of 2 meters rests on supports located at the ends of the beam.



The beam is supporting a load of W kilograms per meter. The resulting deflection y of the beam at a horizontal distance of x meters from the left end can be modeled by

$$A\frac{d^2y}{dx^2} = 2Wx - \frac{1}{2}Wx^2$$

where A is a positive constant.

(a) Solve the differential equation to find the deflection y as a function of the horizontal distance x.

(b) Use a graphing utility to determine the location and value of the maximum deflection.

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Damped Motion **In Exercises 11–14, consider a damped mass-spring system whose motion is described by the differential equation**

$$\frac{d^2y}{dt^2} + 2\lambda\frac{dy}{dt} + \omega^2 y = 0.$$

The zeros of its characteristic equation are

$$m_1 = -\lambda + \sqrt{\lambda^2 - \omega^2}$$

and

$$m_2 = -\lambda - \sqrt{\lambda^2 - \omega^2}.$$

For $\lambda^2 - \omega^2 > 0$, the system is *overdamped;* for $\lambda^2 - \omega^2 = 0$, it is *critically damped;* and for $\lambda^2 - \omega^2 < 0$, it is *underdamped.*

(a) Determine whether the differential equation represents an overdamped, critically damped, or underdamped system.

(b) Find the particular solution that satisfies the initial conditions.

(c) Use a graphing utility to graph the particular solution found in part (b). Explain how the graph illustrates the type of damping in the system.

11. $\dfrac{d^2y}{dt^2} + 8\dfrac{dy}{dt} + 16y = 0$

$y(0) = 1,\ y'(0) = 1$

12. $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + 26y = 0$

$y(0) = 1,\ y'(0) = 4$

13. $\dfrac{d^2y}{dt^2} + 20\dfrac{dy}{dt} + 64y = 0$

$y(0) = 2,\ y'(0) = -20$

14. $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + y = 0$

$y(0) = 2,\ y'(0) = -1$

15. Airy's Equation Consider Airy's equation given in Section 16.4, Exercise 25. Rewrite the equation as

$$y'' - (x - 1)y - y = 0.$$

Then use a power series of the form

$$y = \sum_{n=0}^{\infty} a_n(x - 1)^n$$

to find the first *eight* terms of the solution. Compare your result with that of Exercise 25 in Section 16.4.

16. Chebyshev's Equation Consider **Chebyshev's equation**

$$(1 - x^2)y'' - xy' + k^2y = 0.$$

Polynomial solutions of this differential equation are called *Chebyshev polynomials* and are denoted by $T_k(x)$. They satisfy the recursion equation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

(a) Given that $T_0(x) = 1$ and $T_1(x) = x$, determine the Chebyshev polynomials $T_2(x)$, $T_3(x)$, and $T_4(x)$.

(b) Verify that $T_0(x)$, $T_1(x)$, $T_2(x)$, $T_3(x)$, and $T_4(x)$ are solutions of the given differential equation.

(c) Verify the following Chebyshev polynomials.

$T_5(x) = 16x^5 - 20x^3 + 5x$

$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$

$T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$

17. Bessel's Equation: Order Zero The differential equation $x^2y'' + xy' + x^2y = 0$ is known as **Bessel's equation of order zero.**

(a) Use a power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

to find the solution.

(b) Compare your result with that of the function $J_0(x)$ given in Section 9.8, Exercise 65.

18. Bessel's Equation: Order One The differential equation

$$x^2y'' + xy' + (x^2 - 1)y = 0$$

is known as **Bessel's equation of order one.**

(a) Use a power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

to find the solution.

(b) Compare your result with that of the function $J_1(x)$ given in Section 9.8, Exercise 66.

19. Hermite's Equation Consider **Hermite's equation**

$$y'' - 2xy' + 2ky = 0.$$

(a) Use a power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

to find the solution when $k = 4$. [*Hint:* Choose the arbitrary constants such that the leading term is $(2x)^k$.]

(b) Polynomial solutions of Hermite's equation are called *Hermite polynomials* and are denoted by $H_k(x)$. The general form for $H_k(x)$ can be written as

$$H_k(x) = \sum_{n=0}^{P} \frac{(-1)^n k!(2x)^{k-2n}}{n!(k-2n)!}$$

where P is the greatest integer less than or equal to $k/2$. Use this formula to determine the Hermite polynomials $H_0(x)$, $H_1(x)$, $H_2(x)$, $H_3(x)$, and $H_4(x)$.

20. Laguerre's Equation Consider **Laguerre's equation**

$$xy'' + (1 - x)y' + ky = 0.$$

(a) Polynomial solutions of Laguerre's equation are called *Laguerre polynomials* and are denoted by $L_k(x)$. Use a power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

to show that

$$L_k(x) = \sum_{n=0}^{k} \frac{(-1)^n k! x^n}{(k-n)!(n!)^2}.$$

Assume that $a_0 = 1$.

(b) Determine the Laguerre polynomials $L_0(x)$, $L_1(x)$, $L_2(x)$, $L_3(x)$, and $L_4(x)$.

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